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VV256 RC6

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- Matrix ODE
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Matrix ODE

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \Lambda^2 = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}$$

If A is diagonalizable, then we can write $A = P\Lambda P^{-1}$.

So $A^2 = (P\Lambda P^{-1})(P\Lambda P^{-1}) = P\Lambda^2 P^{-1}$.

$e^A = \Sigma_{k\geq 0}\frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

$= PIP^{-1} + P\Lambda P^{-1} + P\frac{\Lambda^2}{2!}P^{-1} + P\frac{\Lambda^3}{3!}P^{-1} + \cdots$

$= P(I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \cdots)P^{-1}$.

Suppose f has the Taylor expansion: $f(x) = \Sigma_{k\geq 0}\frac{f^{(k)}(0)}{k!}x^k$.

We can define $f(A) = \Sigma_{k\geq 0}\frac{f^{(k)}(0)}{k!}A^k$.

$$\begin{pmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\cdots & & & \\ & 1+\lambda_2+\frac{\lambda_2^2}{2!}+\cdots & & \\ & & \ddots & \\ & & & \sum_{k\geq 0}\frac{\lambda_n^k}{k!} \end{pmatrix}$$

Solve Φ' = AΦ

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

① Eigenvalue $\lambda_1 = 1,\ \lambda_2 = 5$

② $A = P \Lambda P^{-1} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}^{-1}$

$$e^{tA} = I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \cdots$$

$$= \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}^{-1}$$

$\therefore e^{tA}$ is the solution to the ODE $\dot{\Phi} = A\Phi,\ \Phi(0) = I$

$$\Phi(t) = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}^{-1}$$

Solve x' = Ax

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \qquad x' = Ax \qquad x(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{aligned} x(t) &= \Phi(t)\, x(0) \\ &= \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} \\ &= \tilde{c}_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \tilde{c}_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Eigenvalues of Diagonalizable matrix

Suppose A is diagonalizable, with eigenvalues $\lambda_1 \cdots \lambda_n$, i.e.,
$Av_1 = \lambda_1 v_1, \cdots, Av_n = \lambda_n v_n$, then we can write
$A = [v_1, \cdots, v_n]\Lambda[v_1, \cdots, v_n]^{-1}$.

For a vector ODE $x' = Ax$, the general solution is given by:

$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$.

Same as Page 4.

Q1

Solve $x' = Ax$, where A =

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

$x' = Ax$ $\quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$ $\quad \det(A - \lambda I) = -\lambda^3 + 1 + 1 - [-\lambda - \lambda - \lambda] = 0$

$-\lambda^3 + 2 + 3\lambda = 0$

$\lambda^3 - 3\lambda - 2 = 0$ $\quad (\lambda - 2)(\lambda + 1)^2 = 0$ $\quad \lambda_1 = 2 \quad \lambda_2 = \lambda_3 = -1$

$\lambda = 2$ $\quad \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad x_1 = x_2 = x_3 = 1$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = -1$ $\quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$x_2 = x_2$

$x_3 = x_3$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{-t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

$$x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Relationship

Given a vector ODE $x'(t) = A(t)x(t)$, Suppose $\phi(t) = [x^{(1)}(t), x^{(2)}(t), \cdots, x^{(n)}(t)]$, where $x^{(1)}(t), \cdots, x^{(n)}(t)$ form a fundamental solution set to the ODE, than the solution to $\Phi'(t) = A(t)\Phi(t)$ is given by $\Phi(t) = \phi(t)\phi^{-1}(0)$.

Properties

$1+i \Rightarrow 1-i$

$= \bar{A}^T$

(1) All eigenvalues are distinct.

(2) $A = A^T$ for real matrices, and $A = A^*$ for complex matrices.

(3) For a matrix A with distinct n eigenvalues, then A is diagonalizable, i.e., All eigenvectors are linearity independent.

(4) A is diagnoalizable $\Longleftrightarrow AA^* = A^*A$

(5) We say A is similar to B if $A = PBP^{-1}$ for some invertible square matrix P.

Theorem for real matrix

For a real matrix A, TFAE:

(1) $A = A^T$

(2) Exist ONB(Orthogonal Normal Basis) for R^n s.t. it consists of eigenvectors of A.

(3) A is orthogonally diagonalizable, i.e., exist a matrix Q s.t. $QQ^T = I_n$, Λ diagonal matrix s.t. $A = Q\Lambda Q^T$.

EVD (orthogonal form)

Example: Do the Eigenvalue Decomposition for A, where A =

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q2 solution

Orthogonal form:

$$A = Q \Lambda Q^T \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\lambda = 0, 0, 3$$

$$\lambda = 0 \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x = \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$q_2 = \frac{v_2 - q_1 \langle q_1, v_2 \rangle}{\| v_2 - q_1 \langle q_1, v_2 \rangle \|} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} / \sqrt{6} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$\lambda = 3 \qquad q_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{3} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$A = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^T$$

Jordan Normal Form

An $n \times n$ matrix A can be diagonalized only if it has a full complement of n linearly independent eigenvectors. If there is a shortage of eigenvectors (because of repeated eigenvalues), then A can always be transformed into a nearly diagonal matrix called its Jordan form, which has the eigenvalues of A on the main diagonal, ones in certain positions on the diagonal above the main diagonal, and zeros elsewhere.

Q3

Let J =

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

, where λ is an arbitary real number.

(1) Find J^2, J^3, J^4.

(2) Use an inductive argument to find J^n.

(3) Determine e^{Jt}.

(a)

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}$$

(b)

(b). Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\mathbf{J}^{n+1} = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda \cdot \lambda^n & 0 & 0 \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}.$$

Hence the result follows by mathematical induction.

(c)

$$\sum_{n=0}^{\infty} \lambda^n \frac{t^n}{n!} = e^{\lambda t}$$

$$\sum_{n=0}^{\infty} n\lambda^{n-1} \frac{t^n}{n!} = t \sum_{n=1}^{\infty} \lambda^{n-1} \frac{t^{n-1}}{(n-1)!} = te^{\lambda t}$$

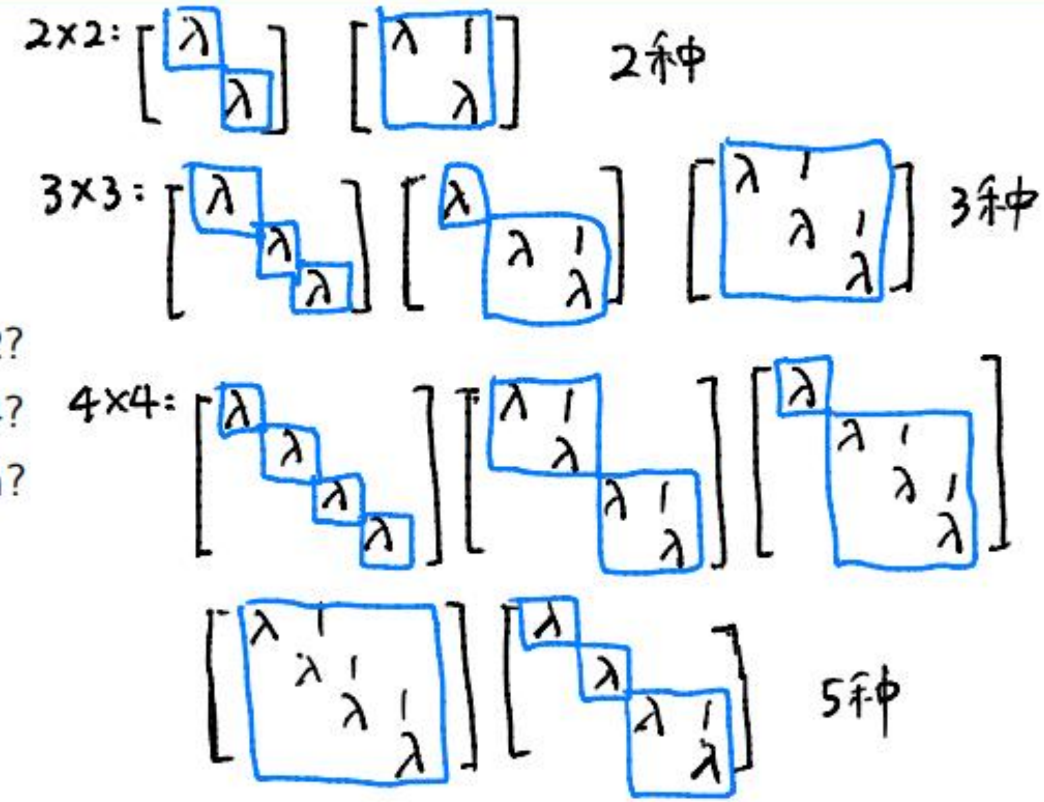
$$\exp(\mathbf{J}t) = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$$

Form of J

(1) 2*2?

(2) 4*4?

(3) n*n?



Q4

Find the Jordan Normal Form for A, where A =

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = 2$

$[A - 2I_3] V_1 = 0 \rightarrow V_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$[A - 2I_3] V_2 = V_1 \rightarrow V_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

$[A - 2I_3] V_3 = V_2 \rightarrow V_3 = \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}$

Jordan Normal Form

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Q5

$A v_1 = 1 \cdot v_1$

$A v_2 = 1 \cdot v_2$

$A v_3 = 1 \cdot v_3 + v_2$

Find the Jordan Normal Form for A, where A =

$$\begin{bmatrix} 5 & 3 & -2 \\ 8 & -5 & -4 \\ 4 & 3 & 3 \end{bmatrix}$$

$$JNF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3$

$(A - I_3)v = 0$

$$v = c_1 \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

geometric multiplicity is 2

$$v_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \quad v_2 = \alpha \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$(A - I_3)v_3 = \begin{bmatrix} 3\alpha+1 \\ 4\alpha \\ 2 \end{bmatrix} \Rightarrow \alpha = -1$$

$$v_2 = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}$$

Q6 (Very Difficult)

Given a real matrix A, where A =

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(1) Find the Jordan Normal Form of A.

(2) Find the general solution to the ODE $x' = Ax$.

Solution:

Solution: Since A is upper triangular, we know that the eigenvalues are $\lambda = 0, 1, 2$.
When $\lambda = 0$, we solve $(A - 0I)u_1 = u_1$, so $u_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$ is an eigenvector for $\lambda = 0$.
When $\lambda = 1$, we solve $(A - 1I)v_1 = v_1$, i.e.,

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} v_1 = 0 \tag{24}$$

then $v_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$ is an eigenvector for $\lambda = 1$.
When $\lambda = 2$, we solve $(A - 1I)w_1 = w_1$, i.e.,

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} w_1 = 0 \tag{25}$$

then $w_1 = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 0 \end{bmatrix}^\top$ is an eigenvector for $\lambda = 2$.
At this stage, we know that the geometric multiplicity of $0, 1, 2$ are all equal to 1. We can thus conclude that the Jordan normal form is

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{26}$$

(order of individual Jordan block may vary)

Solution: For $\lambda = 1$, we need to find generalized eigenvectors v_2, v_3 such that $Av_2 = v_2 + v_1$, and $Av_3 = v_3 + v_2$. First solve $(A - I)v_2 = v_1$, i.e.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

We get a particular solution $v_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\top$. Similarly we solve $(A - I)v_3 = v_2$, and get a particular solution $v_3 = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}^\top$.
For $\lambda = 2$, we also need to find another generalized eigenvector w_2 such that $Aw_2 = 2w_2 + w_1$. Solve $(A - 2I)w_2 = w_1$, and we get a particular solution $w_2 = \begin{bmatrix} 0 & -5 & -2 & -1 & 0 & \frac{1}{2} \end{bmatrix}^\top$. Now note that we can write $A = PJP^{-1}$, where

$$P = \begin{bmatrix} u_1 & v_1 & v_2 & v_3 & w_1 & w_2 \end{bmatrix} \tag{28}$$

Note that $e^{tA} = Pe^{tJ}P^{-1}$, and

$$e^{tJ} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^t & te^t & \frac{t^2}{2}e^t & 0 & 0 \\ 0 & 0 & e^t & te^t & 0 & 0 \\ 0 & 0 & 0 & e^t & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & 0 & 0 & e^{2t} \end{bmatrix} \tag{29}$$

Therefore the general solution is given by $x(t) = Pe^{tJ}c$, where $c = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}^\top \in \mathbb{R}^6$ is a constant vector. That is

$$\begin{aligned} x(t) &= \begin{bmatrix} u_1 & v_1 & v_2 & v_3 & w_1 & w_2 \end{bmatrix} \begin{bmatrix} c_1 \\ e^t(c_2 + c_3 t + c_4 \frac{t^2}{2}) \\ e^t(c_3 + c_4 t) \\ e^t c_4 \\ e^{2t}(c_5 + c_6 t) \\ e^{2t} c_6 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e^t(c_2 + c_3 t + c_4 \tfrac{t^2}{2}) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e^t(c_3 + c_4 t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &\quad + c_4 e^t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e^{2t}(c_5 + c_6 t) \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_6 e^{2t} \begin{bmatrix} 0 \\ -5 \\ -2 \\ -1 \\ 0 \\ 1/2 \end{bmatrix} \end{aligned} \tag{30}$$

with $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$.

Another method

https://blog.csdn.net/lafea/article/details/134916409
我找不到我当年看的知乎链接了
搜索关键词"代数几何重数"+"若当矩阵"

Jordan 标准型

① 特征值 J的主对角元 $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$

② 特征值 λ_i 的代数重数 (重根数) J上 λ_i 出现次数

↓满足 $p_1 + p_2 + \cdots + p_i = n$

③ 特征值 λ_i 的几何重数 (主对角元为 λ_i 的 Jordan块个数)

$q_{\lambda_i} = n - \text{rank}(A - \lambda_i I)$

一个Jordan块 ⇔ 一个独立特征向量 / 一个几何重数

eg: $A=\begin{bmatrix} 2 & 3 & 2 \\ 1 & 8 & 2 \\ -2 & -14 & -3 \end{bmatrix}$

$f(\lambda)=|\lambda I-A|=\begin{vmatrix} \lambda-2 & -3 & -2 \\ -1 & \lambda-8 & -2 \\ 2 & 14 & \lambda+3 \end{vmatrix}=(\lambda-1)(\lambda-3)^2$

$\lambda_1=1 \quad p_1=1 \quad q_1=n-\mathrm{rank}(A-I)=3-2=1$

$\begin{vmatrix} -1 & -3 & -2 \\ -1 & -7 & -2 \\ 2 & 14 & 4 \end{vmatrix}$

1个1阶的Jordan块

$\lambda_2=\lambda_3=3$

$\begin{vmatrix} 1 & -3 & -2 \\ -1 & -5 & -2 \\ 2 & 14 & 6 \end{vmatrix}$

$\begin{vmatrix} -1 & 3 & 2 \\ 1 & 5 & 2 \\ -2 & -14 & -6 \end{vmatrix} \rightarrow \begin{vmatrix} -1 & 3 & 2 \\ 0 & -4 & -2 \\ -2 & -14 & -6 \end{vmatrix}$

$\mathrm{rank}(A-\lambda I)=2 \quad q_2=1$

2 10 4

1个2阶的Jordan块

$\begin{vmatrix} -2 & 6 & 4 \\ 0 & 0 & 0 \\ -2 & -14 & -6 \end{vmatrix} \leftarrow \begin{vmatrix} -2 & 6 & 4 \\ 0 & -20 & -10 \\ -2 & -14 & -6 \end{vmatrix}$

$\rightarrow J=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ or $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

cool ✓

然后再反求特征向量矩阵（但不好算）

考虑更推荐前种

Thank You !

From Cyberpunk 2077