JOINT INSTITUTE
交大密西根学院

UM-SJTU JOINT INSTITUTE

VV256 RC5

Shao Yujie

Contents

- Matrix ODE
- Linear Algebra
- Eigenvalue and Eigenvector
- EVD
- SVD (optional)

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Matrix ODE

Objective of Matrix ODE

Given an n-th order linear homogeneous ODE: $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = 0$

We define that $x_1 = y$, $x_2 = y'$, $\cdots x_n = y^{(n-1)}$.

So we can transform the ODE into the matrix form, and what we need to solve is $x' = Ax$.

Linear Algebra

Sorry for that I haven’t taken the VE214 and my linear algebra is poor and I cannot explain this part at a higher and more comprehensive level. Also I am too lazy to study again, so I just show you how to use these lemmas without understanding the true meanings. My RC for linear algebra part is more like the calculating exercises instead of math itself.

If you want to study true math, you can ask another TA like Huang jiayue or Chen Yifan. They have better understanding about this part and I am the worst one. If you ask me questions like “how to understand XX”, I will laugh at you and consider it as useless things.

If you want to further learning about linear algebra, you are welcomed to DD MATH and ask questions to 大数学家.

Determinant

https://blog.csdn.net/weixin_46664967/article/details/113621821

Determinant of a n by n matrix

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

Can be calculated as $det(A) = a_{i1}A_{i1} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij}$, where M_{ij} is the Algebraic cofactor of the a_{ij}.
Cofactor: In the determinant of order n, the determinant of order n-1 formed by deleting the elements in row i and column j where element aij is located, is called the cofactor of element a_{ij}.

Laplace Expansion

例：$\begin{vmatrix} 0 & 1 & 0 \\ 2 & 15 & 3 \\ 1 & 41 & 2 \end{vmatrix} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 1$

以三阶行列式为例：

例：$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

以第一行展开，得$D_3 =$

$$= (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$$

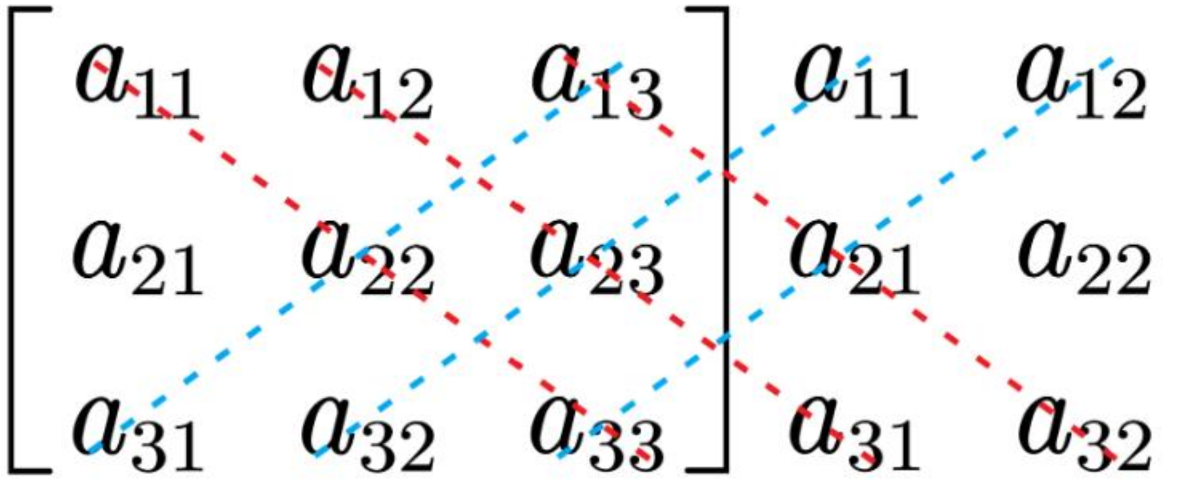
$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

对于**任一行（列）都可进行展开**

以三阶行列式为例:

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

①将第一、二列平移到行列式右侧
②如图做出六条斜对角线
③对角线上的元素**相乘**，**红色**相加的和 **减去** **蓝色**相加的和



Lemma maybe used

Sarrus formula: 用于快速计算3*3矩阵行列式

给定一个 3×3 矩阵：

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

它的行列式（$\det(A)$）可以通过以下步骤计算：

A triangular 3 x 3 matrix is invertible if and only if all its diagonal entries are nonzero. (Proof by Sarrus formula)

步骤：

1. **将矩阵的前两列抄写到右侧**，构造一个扩展矩阵：

$$\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

2. **从左上到右下计算三条对角线的乘积并相加**：

$$(a \cdot e \cdot i) + (b \cdot f \cdot g) + (c \cdot d \cdot h)$$

3. **从左下到右上计算三条对角线的乘积并相加**：

$$(g \cdot e \cdot c) + (h \cdot f \cdot a) + (i \cdot d \cdot b)$$

4. **用第2步的结果减去第3步的结果**：

$$\det(A) = [(a \cdot e \cdot i) + (b \cdot f \cdot g) + (c \cdot d \cdot h)] - [(g \cdot e \cdot c) + (h \cdot f \cdot a) + (i \cdot d \cdot b)]$$

对于矩阵：

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

1. **扩展矩阵**：

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{bmatrix}$$

2. **从左上到右下的乘积和**：

$$(1 \cdot 5 \cdot 9) + (2 \cdot 6 \cdot 7) + (3 \cdot 4 \cdot 8) = 45 + 84 + 96 = 225$$

3. **从左下到右上的乘积和**：

$$(7 \cdot 5 \cdot 3) + (8 \cdot 6 \cdot 1) + (9 \cdot 4 \cdot 2) = 105 + 48 + 72 = 225$$

4. **计算行列式**：

$$\det(A) = 225 - 225 = 0$$

Cramer's Rule

5. Cramer's Rule: $A\vec{x} = \vec{b}$. linear system.

$n \times n$.

det $\Leftrightarrow$ Invertible. $\rightarrow$ A: ith column $\Rightarrow \vec{b}$.

$$x_i = \frac{\det(A_{\vec{b},i})}{\det A}.$$

Properties

1. $det(A^T) = det(A)$
2. We can extract the common factor of certain row or column

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

3. decomposition of determinant

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{m}
\end{vmatrix} \text{(the } ith \text{ row)}
$$

$$
= \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
b_1 & b_2 & \cdots & b_n \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} + \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
c_1 & c_2 & \cdots & c_n \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}.
$$

4. interchange between row

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} = - \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

怎么求逆矩阵

$[A|I]$ $[A^{-1}A|A^{-1}I]$ $[I|A^{-1}]$

① 高斯若当

$[A|I] \Leftrightarrow [I|A^{-1}]$ 基本行运算

$$A=\begin{bmatrix}3&0&2\\2&0&-2\\0&1&1\end{bmatrix}$$

$$A|I=\left[\begin{array}{ccc|ccc}3&0&2&1&0&0\\2&0&-2&0&1&0\\0&1&1&0&0&1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}5&0&0&1&1&0\\2&0&-2&0&1&0\\0&1&1&0&0&1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{5}&\frac{1}{5}&0\\2&0&-2&0&1&0\\0&1&1&0&0&1\end{array}\right]$$

×2 减

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{5}&\frac{1}{5}&0\\0&0&-2&-\frac{2}{5}&\frac{3}{5}&0\\0&1&1&0&0&1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{5}&\frac{1}{5}&0\\0&0&1&\frac{1}{5}&-\frac{3}{10}&0\\0&1&1&0&0&1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}1&0&0&0.2&0.2&0\\0&1&1&0&0&1\\0&0&1&0.2&-0.3&0\end{array}\right]$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0.2&0.2&0\\-0.2&0.3&1\\0.2&-0.3&0\end{bmatrix}$$

$I \cdot A^{-1}$

$$A^{-1}=\begin{bmatrix}0.2&0.2&0\\-0.2&0.3&1\\0.2&-0.3&0\end{bmatrix}$$

2) 代数余子式　　太烦了，你们自己去搜

余子式、代数余子式和伴随来求逆矩阵，计算过程比较繁琐。有以下步骤：

1、求余子式矩阵

2、转成代数余子式矩阵

3、转成伴随矩阵

4、乘以 1/行列式

例子：求 A 的逆矩阵。解：

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

余子式矩阵（计算行列式）：

$$\begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0\times1 - (-2)\times1 = 2$$

$$\begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2\times1 - (-2)\times0 = 2$$

$$\cdots$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3\times-2 - 2\times2 = -10$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3\times0 - 0\times2 = 0$$

$$\begin{bmatrix} 0\times1-(-2)\times1 & 2\times1-(-2)\times0 & 2\times1-0\times0 \\ 0\times1-2\times1 & 3\times1-2\times0 & 3\times1-0\times0 \\ 0\times(-2)-2\times0 & 3\times(-2)-2\times2 & 3\times0-0\times2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$

余子式矩阵

代数余子式矩阵（加入相隔正负号）：

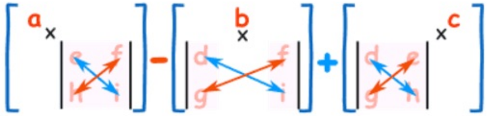
$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}$$

余子式矩阵　　　　余因子矩阵

转成伴随矩阵（转置）：

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

乘以 1/行列式：



矩阵A的行列式 = 3×2 - 0×2 + 2×2 = 10

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

3) 如果脑子记不住

那就待定系数重走来时路

Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is called an eigenvalue of A if there is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The vector $\mathbf{x}$ is called an eigenvector of A corresponding to λ.

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that

$$\det(\lambda I - A) = 0.$$

2. The eigenvectors of A corresponding to λ are the nonzero solutions of

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

Eigenvalue Decomposition

How to judge whether a matrix is diagonalizable or not?

If # of eigenvalue is identical to the dimensionality of that matrix, then the matrix is diagonalizable.

Otherwise, it is not diagonalizable.

Then, if a matrix is diagonalizable, we can rewrite it in the following form:

$$\Sigma = Q\Lambda Q^\top = Q \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p \end{bmatrix} Q^\top$$

$\Sigma Q = Q\Lambda$, so $\Sigma q_j = \lambda_j q_j$. The column vectors in Q are eigenvectors. The diagonal elements in Λ are eigenvalues of Σ.

Q0:

$$A = Q\Sigma Q^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -1-\lambda & 1 & 0 \\ -4 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix}$$

$$= (-1)^{3+3}(2-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (2-\lambda)(\lambda-1)^2$$

$\lambda_1 = 2$

$\lambda_2 = \lambda_3 = 1$

$$(A-2E) = \begin{pmatrix} -3 & 1 & 0 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad p_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(A-E) = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad p_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Q1: ~~有重根~~

$$A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 6-\lambda & 2 & 0 \\ 2 & 6-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$(-1)^{3+3}(3-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(6-\lambda)^2 - 2^2\right] = 0$$

$\lambda = 3$ $\lambda = 4$ $\lambda = 8$

$$\lambda=8 \quad \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$-2x_1+2x_2=0$$
$$2x_1-2x_2=0$$
$$-5x_3=0$$

$$v_1=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda=4 \quad \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$2(x_1+x_2)=0$$
$$-x_3=0$$

$$v_2=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\lambda=3 \quad \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$3x_1+2x_2=0$$
$$2x_1+3x_2=0$$
$$0x_3=0$$

$$v_3=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A=SBS^{-1} \qquad S=\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B=\begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Q2：（局限性）

牢蔡可能是不讲的，但是这个不学，好像不太合理，后续解释jordan normal form会快一点

代数重数/几何重数

https://zhuanlan.zhihu.com/p/618571314

代数重数(algebraic multiplicity)

> 矩阵的特征多项式中某个特征值的重复次数

对于一个矩阵A的特征多项式:

$$f(\lambda) = (\lambda - \lambda_1)^{r_1} + (\lambda - \lambda_2)^{r_2} + \cdots + (\lambda - \lambda_s)^{r_s},\ \sum_{i=1}^{s} r_i = n$$

λ_i **的代数重数就是**$\mathbf{r_i}\ (\mathbf{i} = \mathbf{1}, \mathbf{2}, \cdots, \mathbf{s})$

几何重数(geometric multiplicity)

> 特征值构成的$(A-\lambda I)$矩阵的零空间的维数

e.g.

求下面这个矩阵的**代数重数**和**几何重数**:

$$A=\begin{pmatrix}1&1&2\\0&1&2\\0&0&3\end{pmatrix}$$

- 代数重数:

特征多项式为: $f(\lambda)=(\lambda-1)^2(\lambda-3)$

有两个特征值, $\lambda_1=1$, 二重根, 则代数重数为2

$\lambda_2=3$, 一重根, 则代数重数为1

- 几何重数

$$A-I=\begin{pmatrix}0&1&2\\0&0&2\\0&0&2\end{pmatrix}\quad A-3I=\begin{pmatrix}-2&1&2\\0&-2&2\\0&0&0\end{pmatrix}$$

易知$rank(A-I)=2$, 则零空间的维数为1, λ_1的几何重数为1

$rank(A-3I)=2$, 则零空间的维数为1, λ_2的几何重数为1

Q坤：

$$A = \begin{bmatrix} 5 & 4 \\ 0 & 5 \end{bmatrix}.$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 0 & 5-\lambda \end{bmatrix} \qquad (5-\lambda)^2 = 4$$

$$\lambda_1 = 3 \qquad \lambda_2 = 7$$

$$\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

毁了……

$$\begin{bmatrix} -2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

SVD （optional）

特性	EVD 分解	SVD 分解
适用矩阵	方阵	任意矩阵
分解形式	$A = PDP^{-1}$	$A = U\Sigma V^T$
特征值/奇异值	特征值在对角矩阵 D	奇异值在对角矩阵 Σ
应用范围	系统稳定性分析、对称矩阵对角化	数据降维（PCA）、矩阵压缩、伪逆求解
正交性	特征向量矩阵 P 不一定正交	U 和 V 是正交矩阵

https://blog.csdn.net/qfikh/article/details/103994319

自己看着玩吧，虽然我们好像不考察，但是没道理这个不学

Thank You !

From Mafia : Definitive Edition