UNIVERSITY OF MICHIGAN

JOINT INSTITUTE
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VV256 RC4

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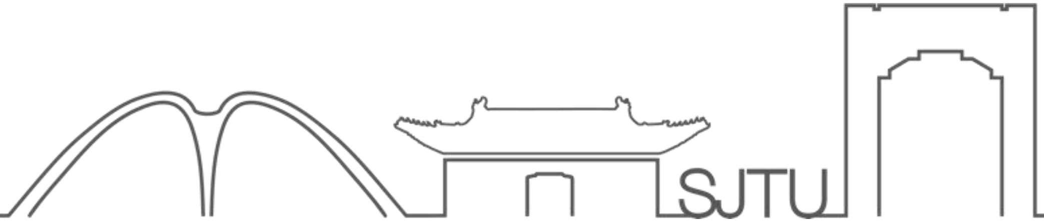


Contents

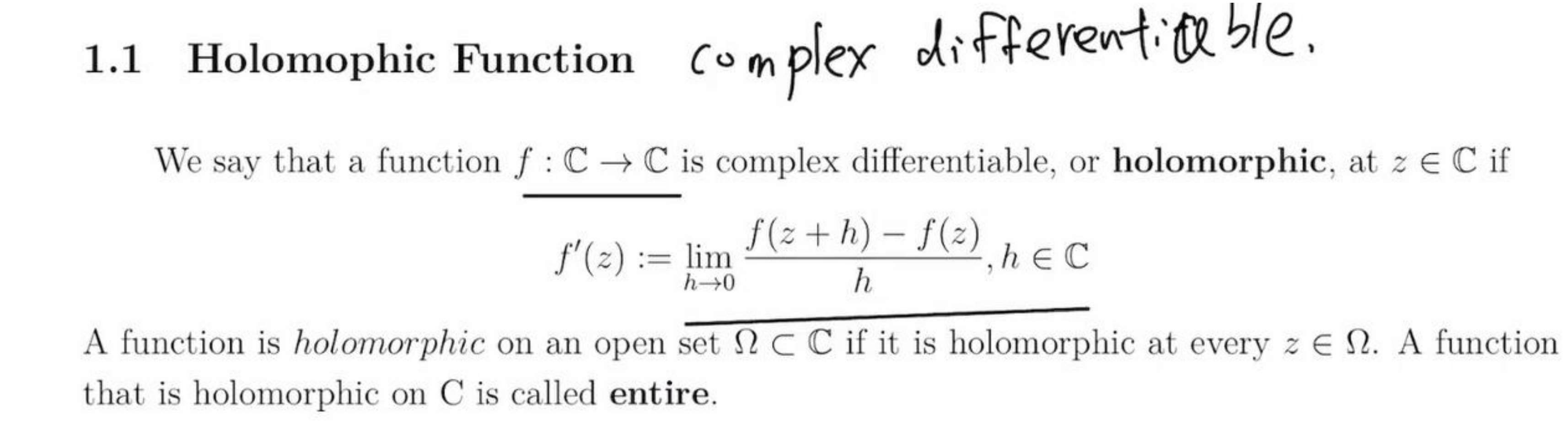
- Holomophic Function
- Cauchy Riemann Equations
- Integration along curve on Complex Plane
- Residue theorem （留数定理）
- 大圆弧/小圆弧

Statement

I have to admit that I cannot understand some logic relationship or use about this part. Indeed, after midterm exam, TAs may fail to know or remember something about the content in the lecture. However, most of us still have the ability to complete the questions in the exam. VV256’s exam is more like a module. If you are famaliar with the example questions and have a strong ability of calculating with a little techniques, the grades will be really nice.

For complex analysis, this part is added to the exam for the first time last year and only TA Wei Linda studied it and made a really perfect RC in Chinese, I remember. So my RC copies many parts from his RC because I have no ideas about how much it will be tested. (I studied too much last year by myself and complex analysis is a very big part and we only cover a little.) If you are not satisfied with my rc, you can ask me to give you his slides (has a little wrong) and his rc recording (very very perfect)

Holomophic Function



1.1 Holomophic Function

complex differentiable.

We say that a function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, or **holomorphic**, at $z \in \mathbb{C}$ if

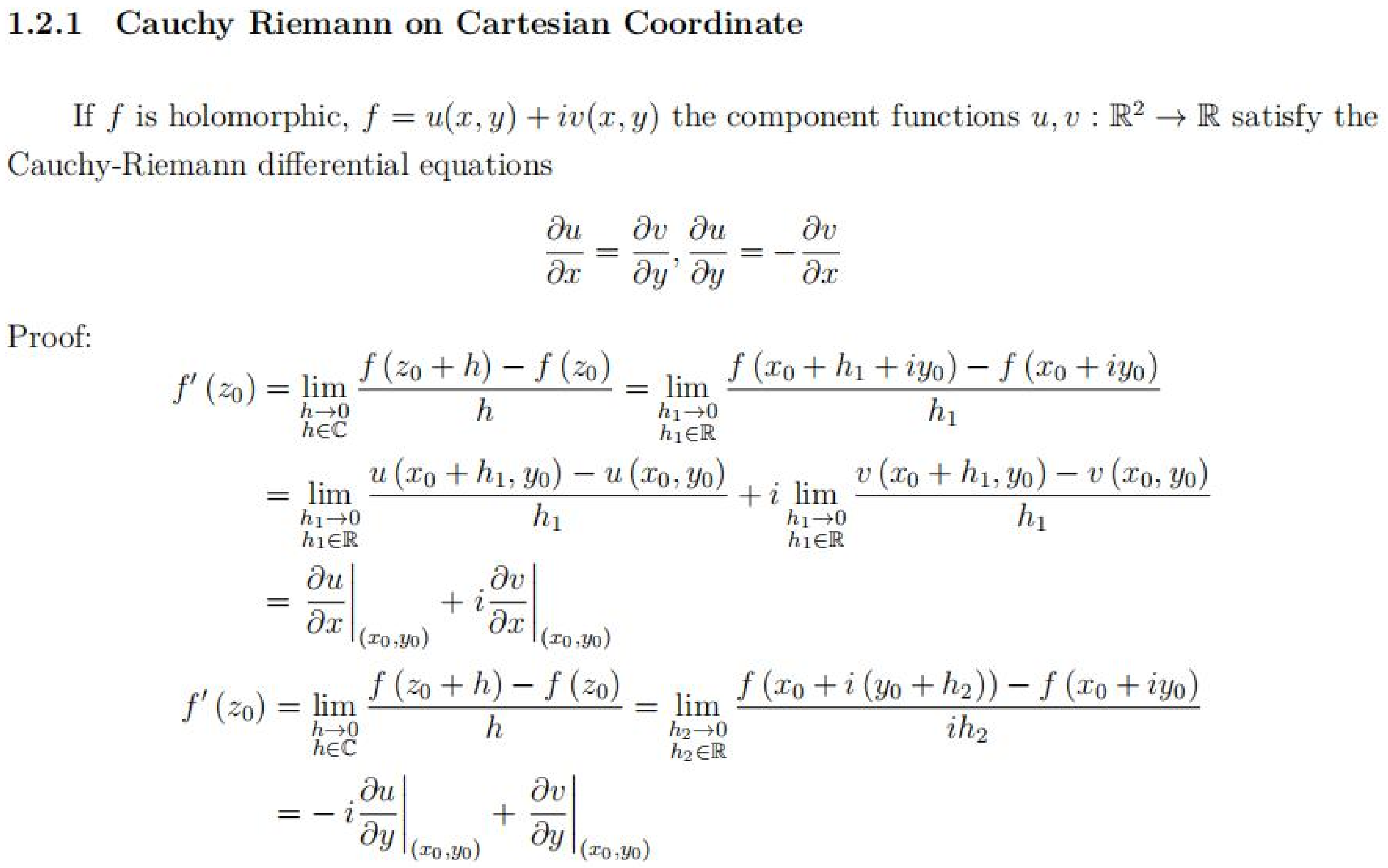
$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, h \in \mathbb{C}$$

A function is *holomorphic* on an open set $\Omega \subset \mathbb{C}$ if it is holomorphic at every $z \in \Omega$. A function that is holomorphic on C is called **entire**.

全纯函数（holomorphic function）是复分析中的一个重要概念，它是指在某个复数域内处处可导的复值函数。全纯函数也被称为解析函数（analytic function），在其定义域内，它不仅具有复数意义下的导数，而且其导数在整个域内是连续的。

全纯函数的定义要求比实值函数更严格，因为全纯函数的导数需要满足柯西-黎曼方程（Cauchy-Riemann equations）。

Cauchy Riemann Equations



1.2.1 Cauchy Riemann on Cartesian Coordinate

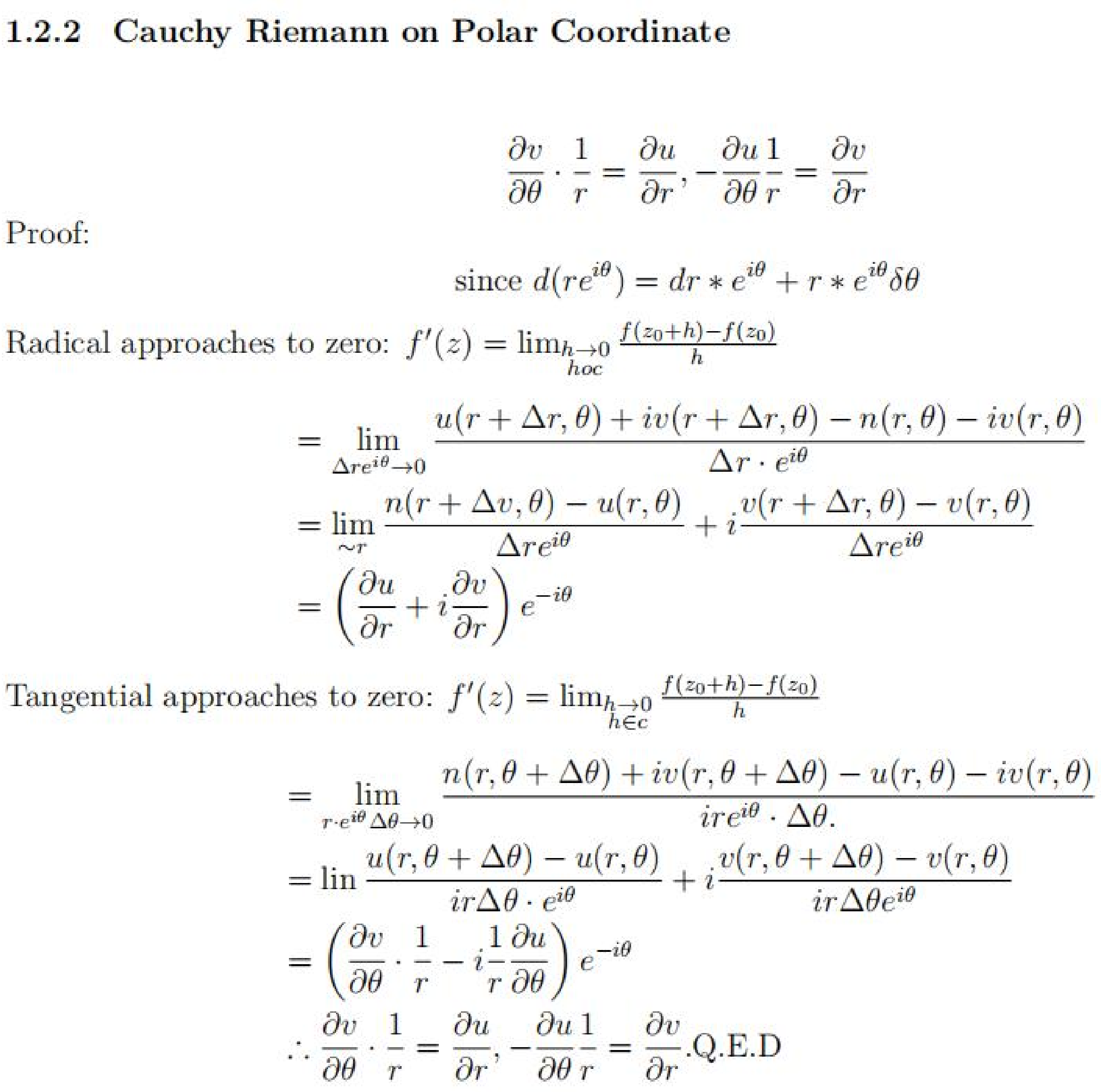
If f is holomorphic, $f = u(x,y) + iv(x,y)$ the component functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof:

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{\substack{h_1\to 0\\ h_1\in\mathbb{R}}} \frac{f(x_0+h_1+iy_0)-f(x_0+iy_0)}{h_1}\\
&= \lim_{\substack{h_1\to 0\\ h_1\in\mathbb{R}}} \frac{u(x_0+h_1,y_0)-u(x_0,y_0)}{h_1} + i \lim_{\substack{h_1\to 0\\ h_1\in\mathbb{R}}} \frac{v(x_0+h_1,y_0)-v(x_0,y_0)}{h_1}\\
&= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}
\end{aligned}$$

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{\substack{h_2\to 0\\ h_2\in\mathbb{R}}} \frac{f(x_0+i(y_0+h_2))-f(x_0+iy_0)}{ih_2}\\
&= -i\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}
\end{aligned}$$



1.2.2 Cauchy Riemann on Polar Coordinate

$$\frac{\partial v}{\partial \theta}\cdot\frac{1}{r}=\frac{\partial u}{\partial r}, -\frac{\partial u}{\partial \theta}\frac{1}{r}=\frac{\partial v}{\partial r}$$

Proof:

$$\text{since } d(re^{i\theta}) = dr * e^{i\theta} + r * e^{i\theta}\delta\theta$$

Radical approaches to zero: $f'(z) = \lim_{\substack{h\to 0\\ hoc}} \frac{f(z_0+h)-f(z_0)}{h}$

$$= \lim_{\Delta re^{i\theta}\to 0} \frac{u(r+\Delta r,\theta)+iv(r+\Delta r,\theta)-n(r,\theta)-iv(r,\theta)}{\Delta r\cdot e^{i\theta}}$$

$$= \lim_{\sim r} \frac{n(r+\Delta v,\theta)-u(r,\theta)}{\Delta re^{i\theta}} + i\frac{v(r+\Delta r,\theta)-v(r,\theta)}{\Delta re^{i\theta}}$$

$$= \left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)e^{-i\theta}$$

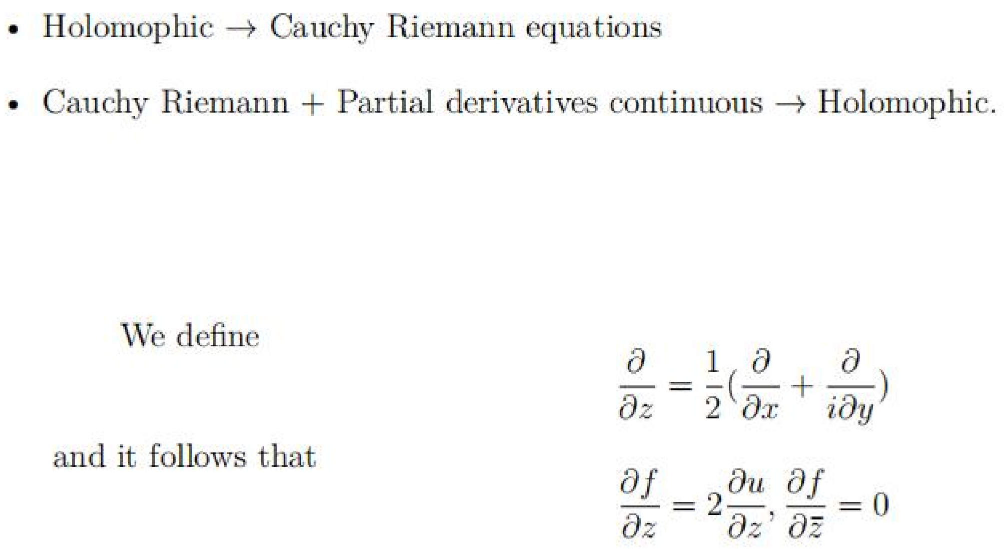
Tangential approaches to zero: $f'(z) = \lim_{\substack{h\to 0\\ h\in c}} \frac{f(z_0+h)-f(z_0)}{h}$

$$= \lim_{r\cdot e^{i\theta}\Delta\theta\to 0} \frac{n(r,\theta+\Delta\theta)+iv(r,\theta+\Delta\theta)-u(r,\theta)-iv(r,\theta)}{ire^{i\theta}\cdot\Delta\theta.}$$

$$= \text{lin}\,\frac{u(r,\theta+\Delta\theta)-u(r,\theta)}{ir\Delta\theta\cdot e^{i\theta}} + i\frac{v(r,\theta+\Delta\theta)-v(r,\theta)}{ir\Delta\theta e^{i\theta}}$$

$$= \left(\frac{\partial v}{\partial \theta}\cdot\frac{1}{r} - i\frac{1}{r}\frac{\partial u}{\partial \theta}\right)e^{-i\theta}$$

$$\therefore \frac{\partial v}{\partial \theta}\cdot\frac{1}{r}=\frac{\partial u}{\partial r}, -\frac{\partial u}{\partial \theta}\frac{1}{r}=\frac{\partial v}{\partial r}\text{.Q.E.D}$$



- Holomophic → Cauchy Riemann equations
- Cauchy Riemann + Partial derivatives continuous → Holomophic.

We define

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{i\partial y}\right)$$

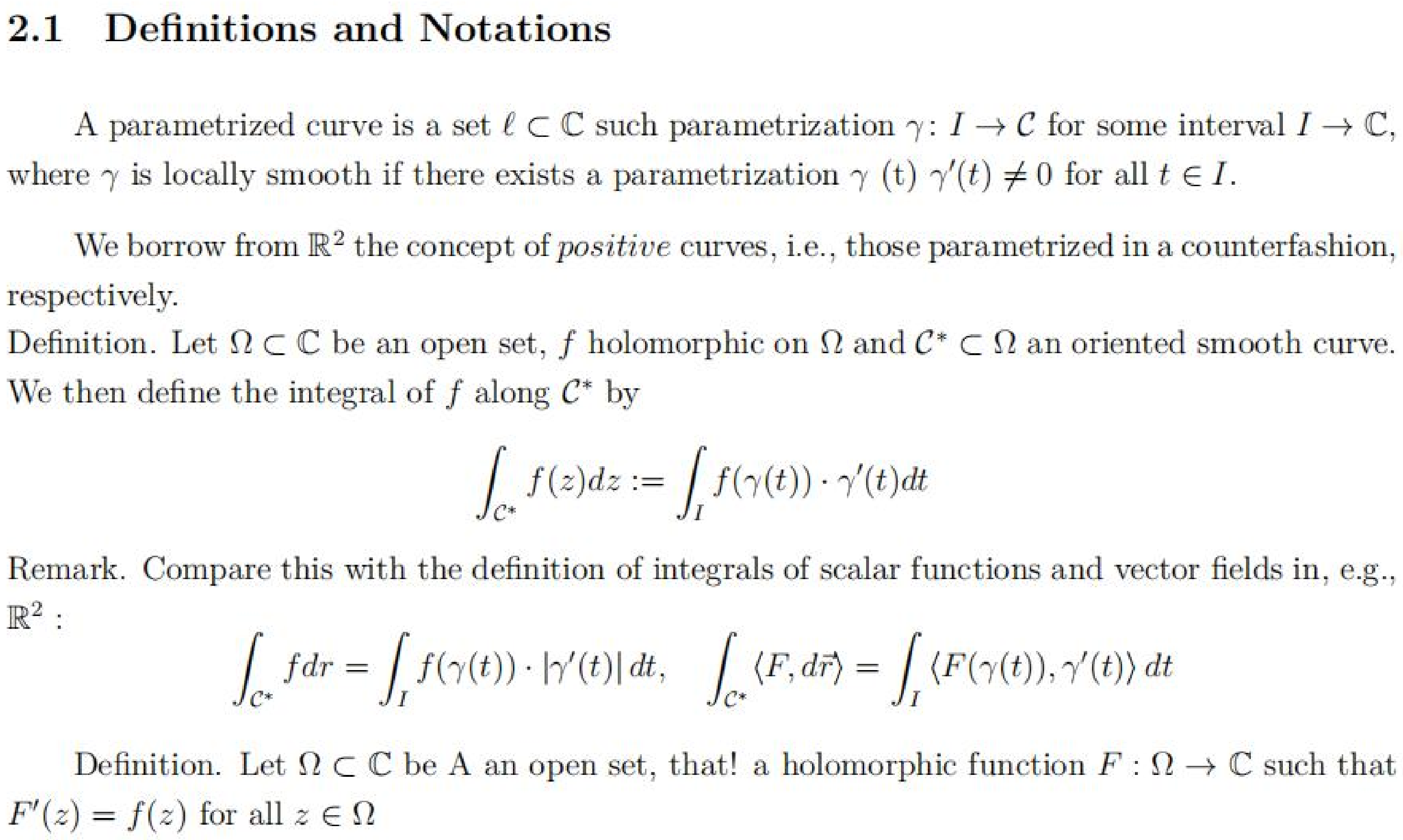
and it follows that

$$\frac{\partial f}{\partial z}=2\frac{\partial u}{\partial z}, \frac{\partial f}{\partial \bar{z}}=0$$

Last year, TA Wei Linda said this part will not be tested and it just helps you understand.

However, I don’t know anything like how much this part will be tested but I guess maybe if you fully understand “留数定理”， it’s enough.

Integration along curve on Complex Plane



2.1 Definitions and Notations

A parametrized curve is a set $\ell \subset \mathbb{C}$ such parametrization $\gamma\colon I \to \mathcal{C}$ for some interval $I \to \mathbb{C}$, where γ is locally smooth if there exists a parametrization γ (t) $\gamma'(t) \neq 0$ for all $t \in I$.

We borrow from $\mathbb{R}^2$ the concept of *positive* curves, i.e., those parametrized in a counterfashion, respectively.

Definition. Let $\Omega \subset \mathbb{C}$ be an open set, f holomorphic on Ω and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve. We then define the integral of f along $\mathcal{C}^*$ by

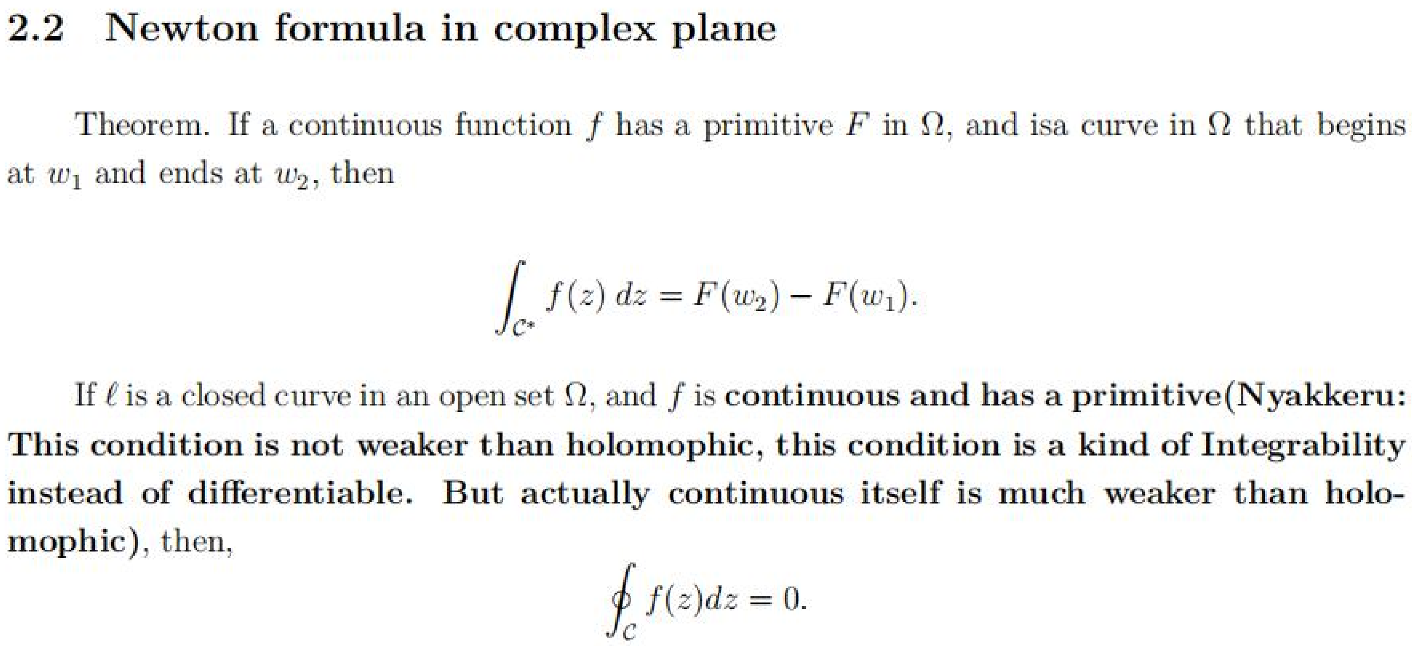
$$\int_{\mathcal{C}^*} f(z)dz := \int_I f(\gamma(t)) \cdot \gamma'(t)dt$$

Remark. Compare this with the definition of integrals of scalar functions and vector fields in, e.g., $\mathbb{R}^2$:

$$\int_{\mathcal{C}^*} f dr = \int_I f(\gamma(t)) \cdot |\gamma'(t)|\, dt, \quad \int_{\mathcal{C}^*} \langle F, d\vec{r} \rangle = \int_I \langle F(\gamma(t)), \gamma'(t) \rangle\, dt$$

Definition. Let $\Omega \subset \mathbb{C}$ be A an open set, that! a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \Omega$

Newton formula in complex plane



2.2 Newton formula in complex plane

Theorem. If a continuous function f has a primitive F in Ω, and isa curve in Ω that begins at w_1 and ends at w_2, then

$$\int_{C^*} f(z)\,dz = F(w_2) - F(w_1).$$

If ℓ is a closed curve in an open set Ω, and f is **continuous and has a primitive(Nyakkeru: This condition is not weaker than holomophic, this condition is a kind of Integrability instead of differentiable. But actually continuous itself is much weaker than holomophic)**, then,

$$\oint_C f(z)dz = 0.$$

复平面里闭合路径环路积分和为0 （very very important）全纯

can be easily understood by physics meaning (保守力场)

but the proof is too difficult (not required)

Steps to Cauchy Theorem

- Goursat's Theorem. Let $\Omega \subset \mathbb{C}$ be open and f holomorphic on Ω. Let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Then

  $$\oint_T f(z)dz = 0.$$

  however I think you do not need to understand it.

- Corollary of Goursat's Theorem. If f is holomorphic in an open an open set Ω that contains a rectangle R and its interior, then

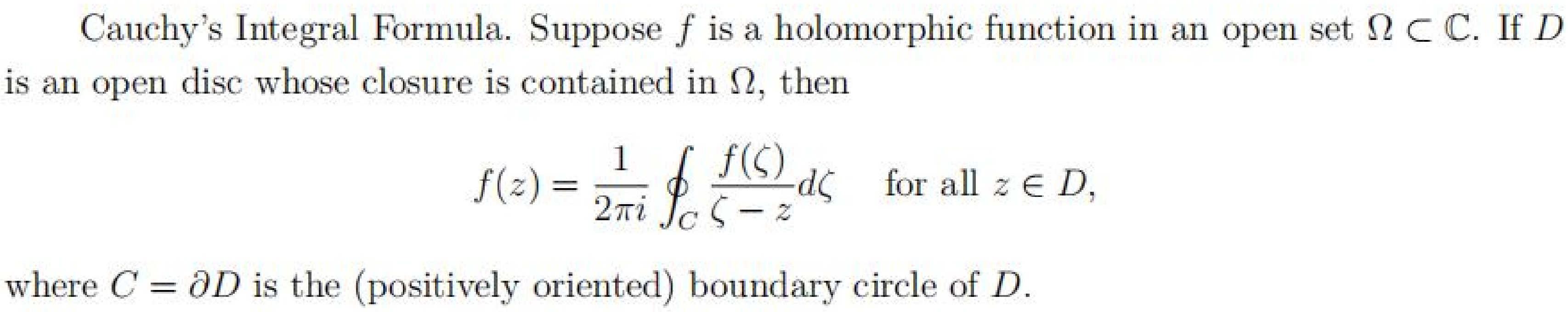
  $$\oint_R f(z)dz = 0$$

- Cauchy Theorem(**Very important**) If f is holomorphic in a disc, then

  $$\oint_C f(z)dz = 0$$

  for any closed curve ℓ in that disc.

Cauchy Integral Formula



Cauchy's Integral Formula. Suppose f is a holomorphic function in an open set $\Omega \subset \mathbb{C}$. If D is an open disc whose closure is contained in Ω, then

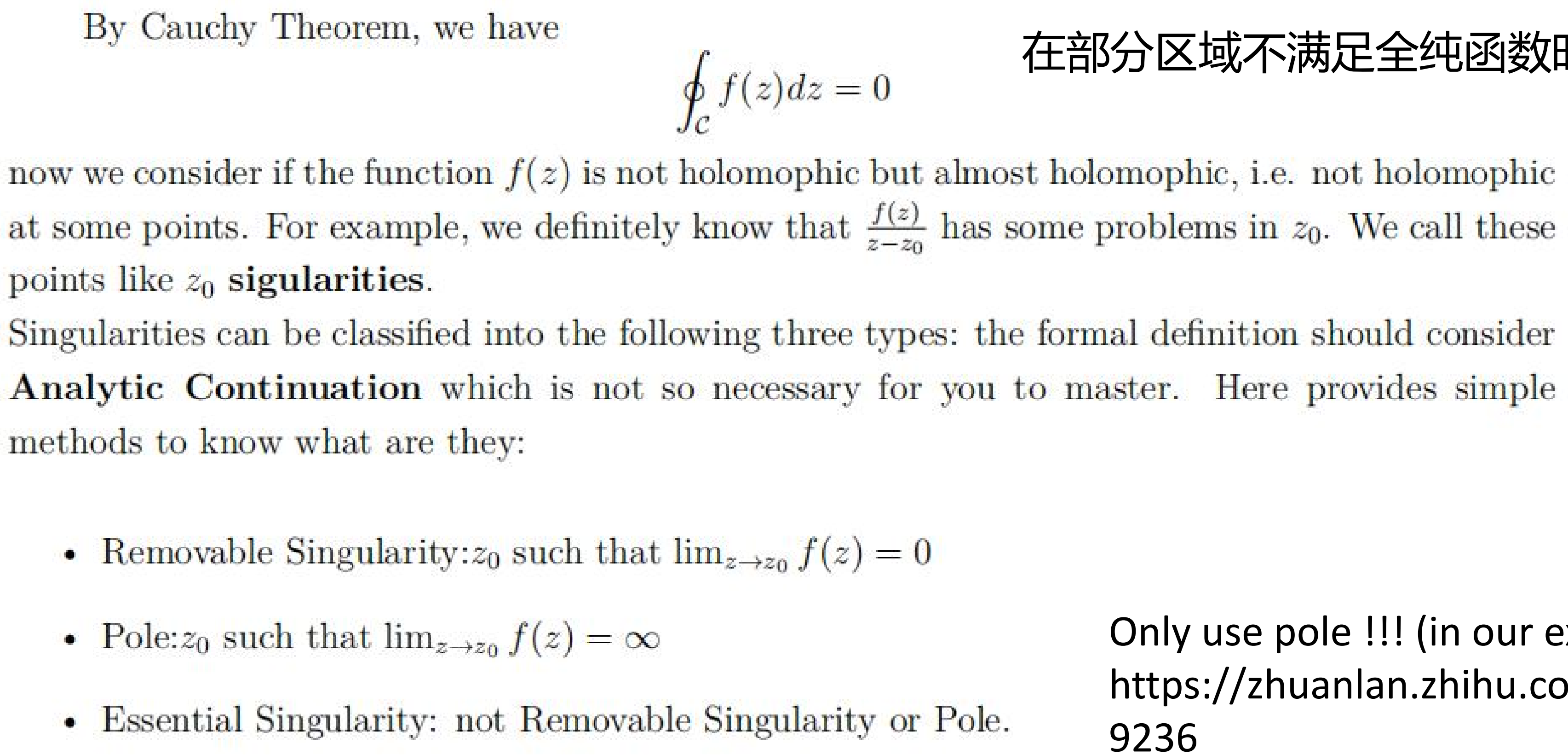
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } z \in D,$$

where $C = \partial D$ is the (positively oriented) boundary circle of D.

If you want to see the proof, you can see this website
https://zhuanlan.zhihu.com/p/129354208

复变函数学习笔记(6) - 知乎 (zhihu.com)

Explanation (by Wei Linda)



By Cauchy Theorem, we have

$$\oint_{\mathcal{C}} f(z)dz = 0$$

now we consider if the function $f(z)$ is not holomophic but almost holomophic, i.e. not holomophic at some points. For example, we definitely know that $\frac{f(z)}{z-z_0}$ has some problems in z_0. We call these points like z_0 **sigularities**.

Singularities can be classified into the following three types: the formal definition should consider **Analytic Continuation** which is not so necessary for you to master. Here provides simple methods to know what are they:

- Removable Singularity:z_0 such that $\lim_{z \to z_0} f(z) = 0$
- Pole:z_0 such that $\lim_{z \to z_0} f(z) = \infty$
- Essential Singularity: not Removable Singularity or Pole.

在部分区域不满足全纯函数时

Only use pole !!! (in our exam)
https://zhuanlan.zhihu.com/p/133309236

Continue :

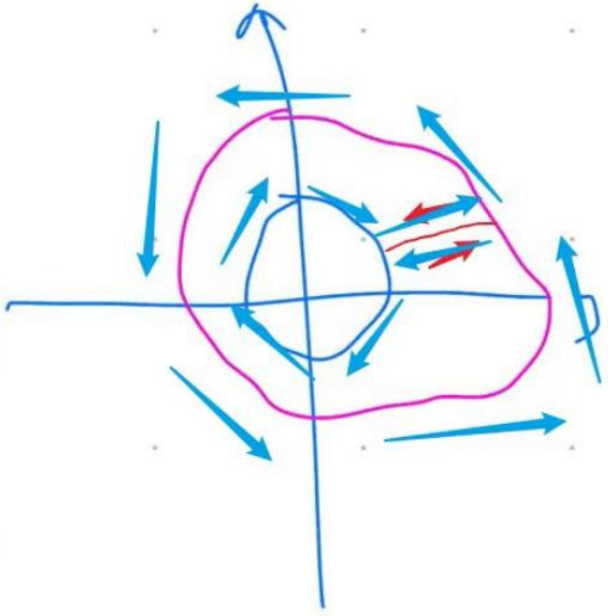


Figure 1: Explain about the last statement

Now I give you insight on how to derive the above theorem. We first consider

$$\oint_C \frac{1}{z} dz = 2\pi i, z_0 = 0$$

in which $\mathcal{C}$ is a closed curve contains z_0.
Why it happens, we suppose C is an unit cycle, i.e.

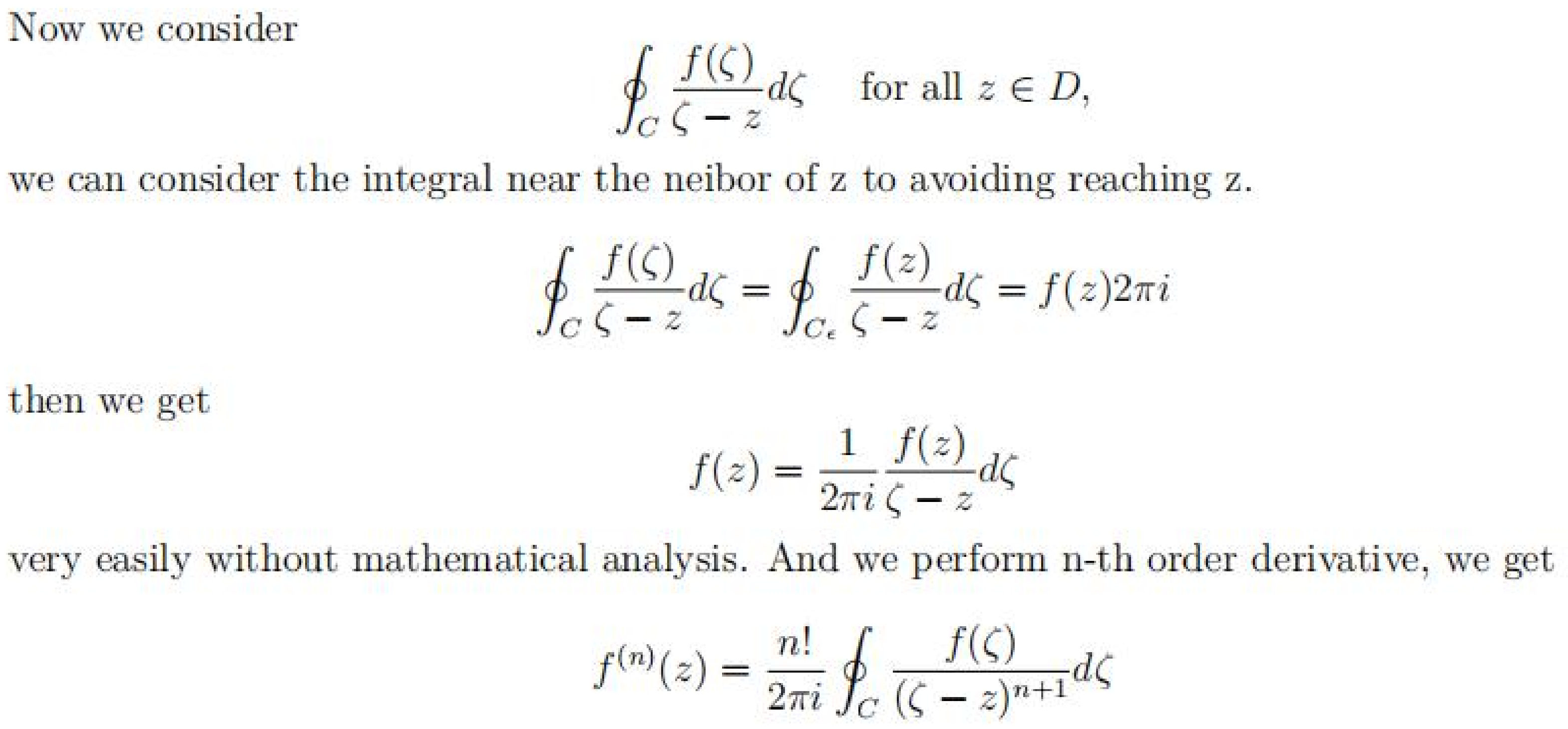
$$\oint_{|z|=1} \frac{1}{z} dz$$

we can change variable using $z = e^{i\theta}$ and we get

$$\int_0^{2\pi} e^{-i\theta} de^{i\theta} = 2\pi i$$

Then, how about closed curve which is not unit circle. Here I will explain why the integral along those curves are the same as the unit circle. By adding a path (red line) to connect the two curve contain the origin(pole in this case) and draw some arrows on the ugly pink and blue circle provided by aa. We can find that the pink and blue curve together with the red path will form a closed curve and the function f is now holomophic on the region formed by the closed curve. Thus if we use this route to integrate, the answer thould be zero.
That is: counterclockwise pink(positive pink)+path(into origin)+clockwise blue(negative blue)+path(out of origin) is zero.
That is: positive pink + negative blue is zero.
That is: pink is blue.
Here, we have shown that:

$$\oint_C \frac{1}{z} dz = 2\pi i$$



Now we consider

$$\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } z \in D,$$

we can consider the integral near the neibor of z to avoiding reaching z.

$$\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = \oint_{C_\epsilon} \frac{f(z)}{\zeta - z} d\zeta = f(z) 2\pi i$$

then we get

$$f(z) = \frac{1}{2\pi i} \frac{f(z)}{\zeta - z} d\zeta$$

very easily without mathematical analysis. And we perform n-th order derivative, we get

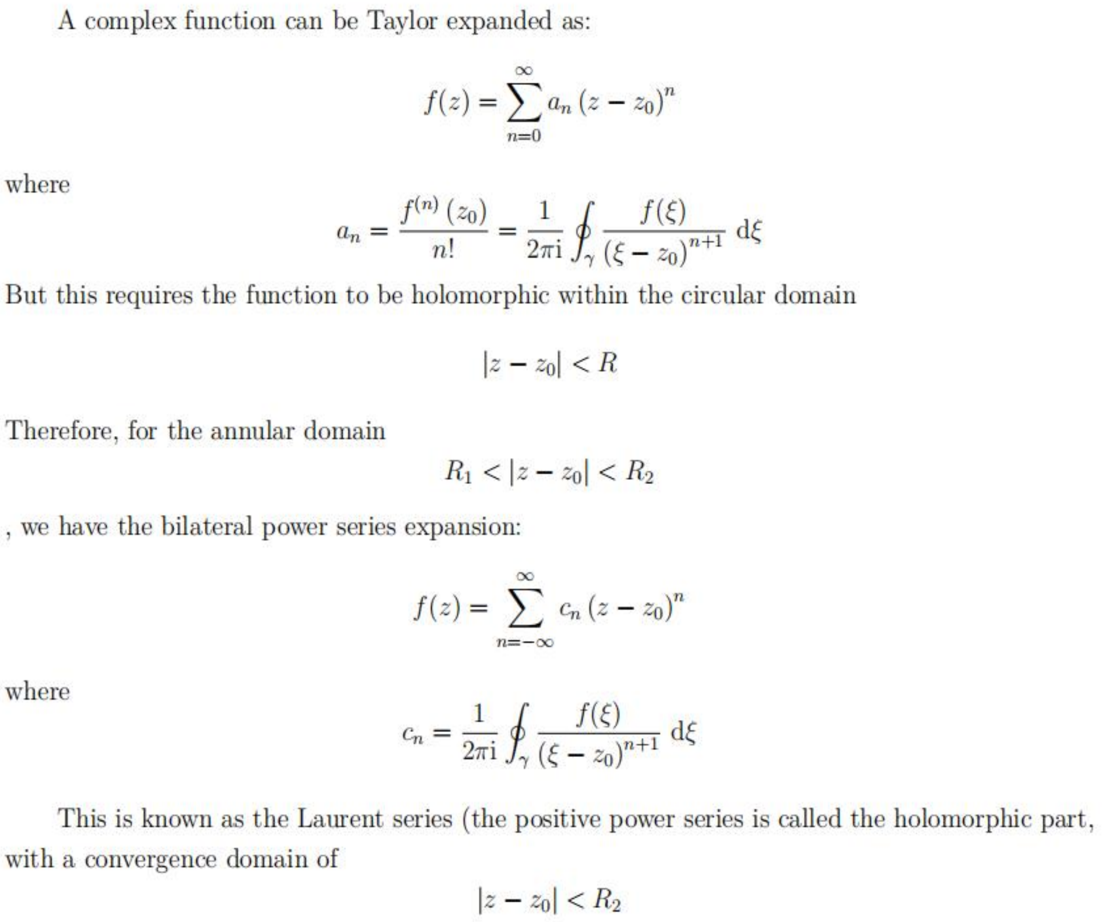
$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Complete !!!

If you cannot fully the logic and relationship like me in the past, it’s ok because the questions are always easy. You just need to remember the Cauchy Integral Formula. It’s enough to solve the exercises and exams.

Laurent Expansion

类似于实函数，一个圆内的复变函数也有Taylor展开式，但是对复变函数而言只要是全纯函数就有Taylor展开式



A complex function can be Taylor expanded as:

$$f(z)=\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$$

where

$$a_n=\frac{f^{(n)}\left(z_0\right)}{n!}=\frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{f(\xi)}{\left(\xi-z_0\right)^{n+1}} \mathrm{d}\xi$$

But this requires the function to be holomorphic within the circular domain

$$\left|z-z_0\right|<R$$

Therefore, for the annular domain

$$R_1<\left|z-z_0\right|<R_2$$

, we have the bilateral power series expansion:

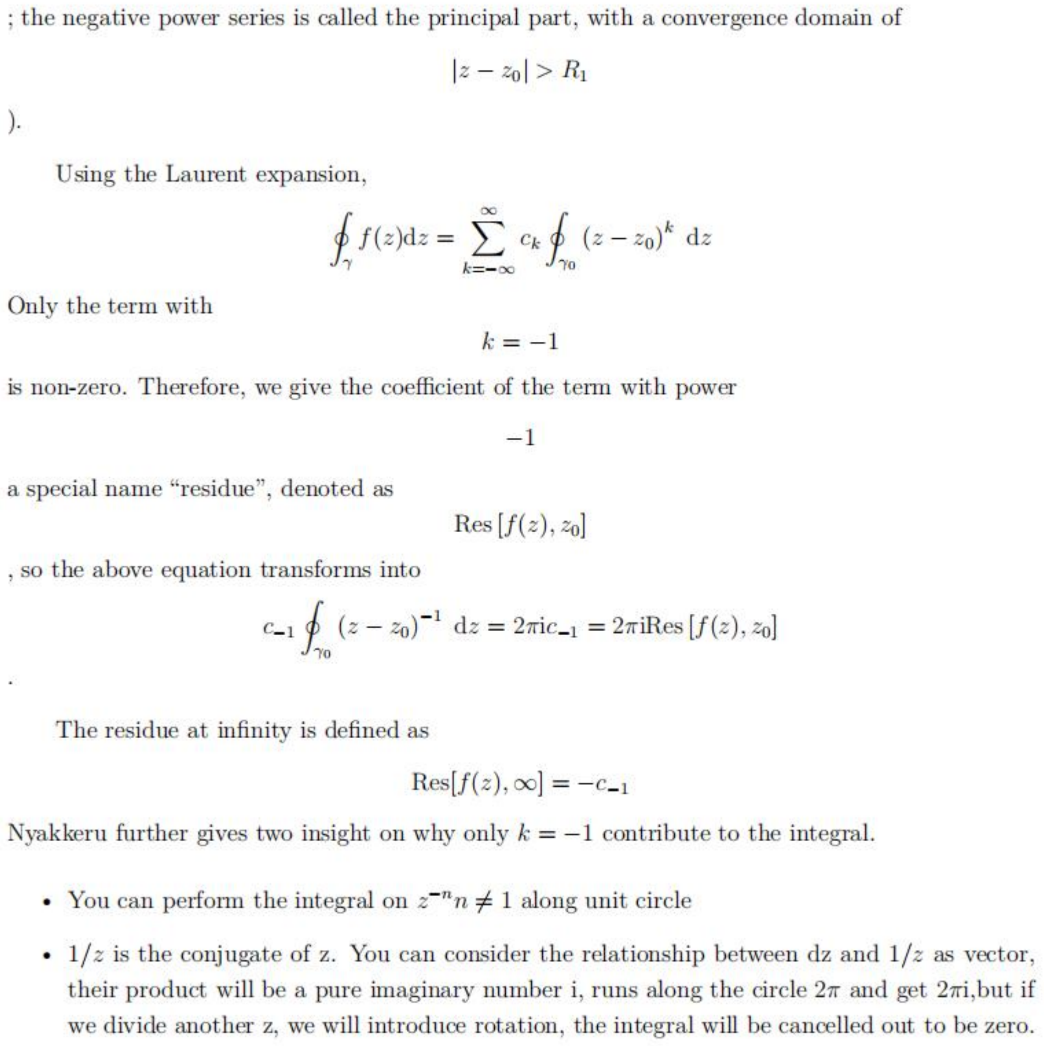
$$f(z)=\sum_{n=-\infty}^{\infty} c_n\left(z-z_0\right)^n$$

where

$$c_n=\frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{f(\xi)}{\left(\xi-z_0\right)^{n+1}} \mathrm{d}\xi$$

This is known as the Laurent series (the positive power series is called the holomorphic part, with a convergence domain of

$$\left|z-z_0\right|<R_2$$



; the negative power series is called the principal part, with a convergence domain of

$$\left|z-z_0\right|>R_1$$

).

Using the Laurent expansion,

$$\oint_\gamma f(z)\mathrm{d}z=\sum_{k=-\infty}^{\infty} c_k \oint_{\gamma_0}\left(z-z_0\right)^k \mathrm{d}z$$

Only the term with

$$k=-1$$

is non-zero. Therefore, we give the coefficient of the term with power

$$-1$$

a special name “residue”, denoted as

$$\operatorname{Res}\left[f(z), z_0\right]$$

, so the above equation transforms into

$$c_{-1} \oint_{\gamma_0}\left(z-z_0\right)^{-1} \mathrm{d}z=2\pi\mathrm{i}c_{-1}=2\pi\mathrm{i}\operatorname{Res}\left[f(z), z_0\right]$$

.

The residue at infinity is defined as

$$\operatorname{Res}[f(z), \infty]=-c_{-1}$$

Nyakkeru further gives two insight on why only $k=-1$ contribute to the integral.

- You can perform the integral on $z^{-n}n \neq 1$ along unit circle
- $1/z$ is the conjugate of z. You can consider the relationship between dz and $1/z$ as vector, their product will be a pure imaginary number i, runs along the circle 2π and get 2πi,but if we divide another z, we will introduce rotation, the integral will be cancelled out to be zero.

Residue Calculus

If you cannot understand the content before, IT'S ALSO OK. You just need to remember the residue calculus theorem and some traditional examples.
https://zhuanlan.zhihu.com/p/133309236

$$\oint_{\gamma} f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname{Res}\left[f(z), z_k\right]$$

is residue theorem.

Laurent Expansion near m-th order pole

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots$$

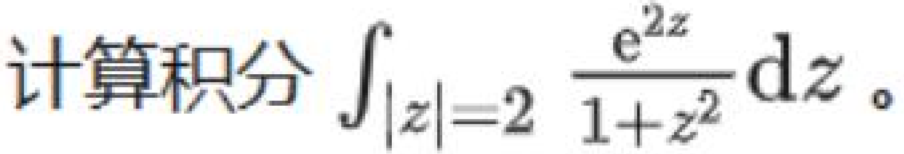
no use

Residue of m-th order pole

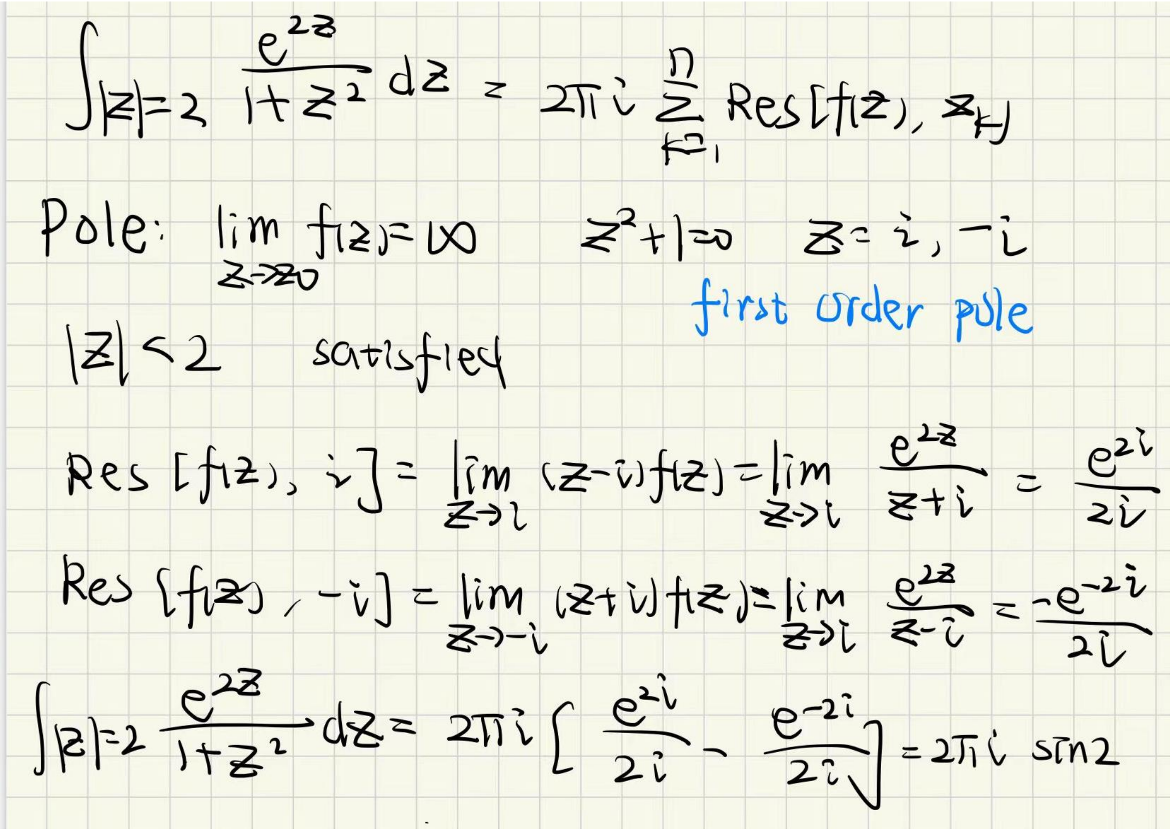
$$\operatorname{Res}\left[f(z), z_0\right] = \frac{1}{(m-1)!}\left[(z-z_0)^m f(z)\right]^{(m-1)}\Big|_{z=z_0}$$

really useful

Q1



计算积分 $\int_{|z|=2}\frac{e^{2z}}{1+z^2}dz$ 。



$$\int_{|z|=2}\frac{e^{2z}}{1+z^2}dz = 2\pi i\sum_{k=1}^{n}\text{Res}[f(z), z_k]$$

Pole: $\lim_{z\to z_0} f(z)=\infty$ $z^2+1=0$ $z=i,-i$

first order pole

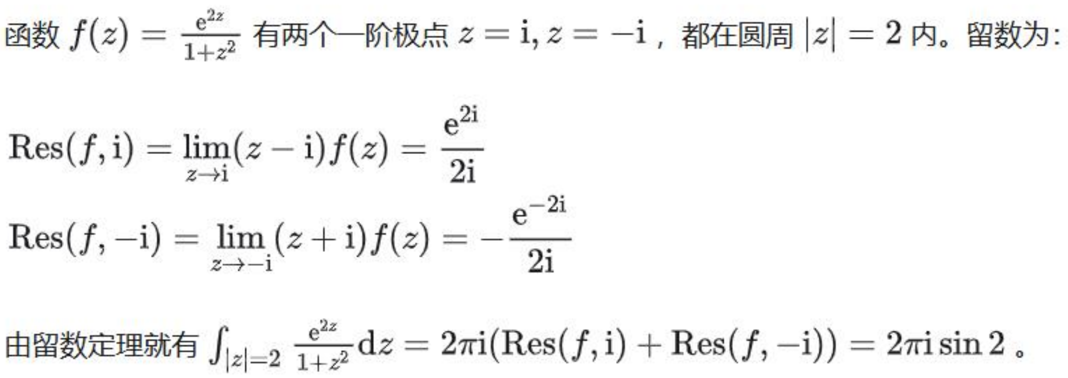
$|z|<2$ satisfied

$$\text{Res}[f(z), i] = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{e^{2z}}{z+i} = \frac{e^{2i}}{2i}$$

$$\text{Res}[f(z), -i] = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to i}\frac{e^{2z}}{z-i} = \frac{-e^{-2i}}{2i}$$

$$\int_{|z|=2}\frac{e^{2z}}{1+z^2}dz = 2\pi i\left[\frac{e^{2i}}{2i}-\frac{e^{-2i}}{2i}\right] = 2\pi i\sin 2$$

First Order Pole



函数 $f(z)=\frac{e^{2z}}{1+z^2}$ 有两个一阶极点 $z=\mathrm{i}, z=-\mathrm{i}$，都在圆周 $|z|=2$ 内。留数为:

$$\text{Res}(f,\mathrm{i}) = \lim_{z\to \mathrm{i}}(z-\mathrm{i})f(z) = \frac{e^{2\mathrm{i}}}{2\mathrm{i}}$$

$$\text{Res}(f,-\mathrm{i}) = \lim_{z\to -\mathrm{i}}(z+\mathrm{i})f(z) = -\frac{e^{-2\mathrm{i}}}{2\mathrm{i}}$$

由留数定理就有 $\int_{|z|=2}\frac{e^{2z}}{1+z^2}dz = 2\pi\mathrm{i}(\text{Res}(f,\mathrm{i})+\text{Res}(f,-\mathrm{i})) = 2\pi\mathrm{i}\sin 2$ 。

Q2 计算积分 $\int_{|z|=2} \frac{1}{z^3(z^{10}-2)}\mathrm{d}z$。

单位根（Root of Unity）是复数中满足 $z^n = 1$ 的复数 z，其中 n 是一个正整数。换句话说，单位根是 1 在复平面上的 n 个等分点。单位根具有周期性和对称性，在复平面上它们构成一个正 n 边形。

具体性质

设 n 是一个正整数，则单位根通常记作 ω_k，其标准形式为：

$$\omega_k = e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k = 0, 1, 2, \ldots, n-1$$

这些 ω_k 都满足：

$$\omega_k^n = 1$$

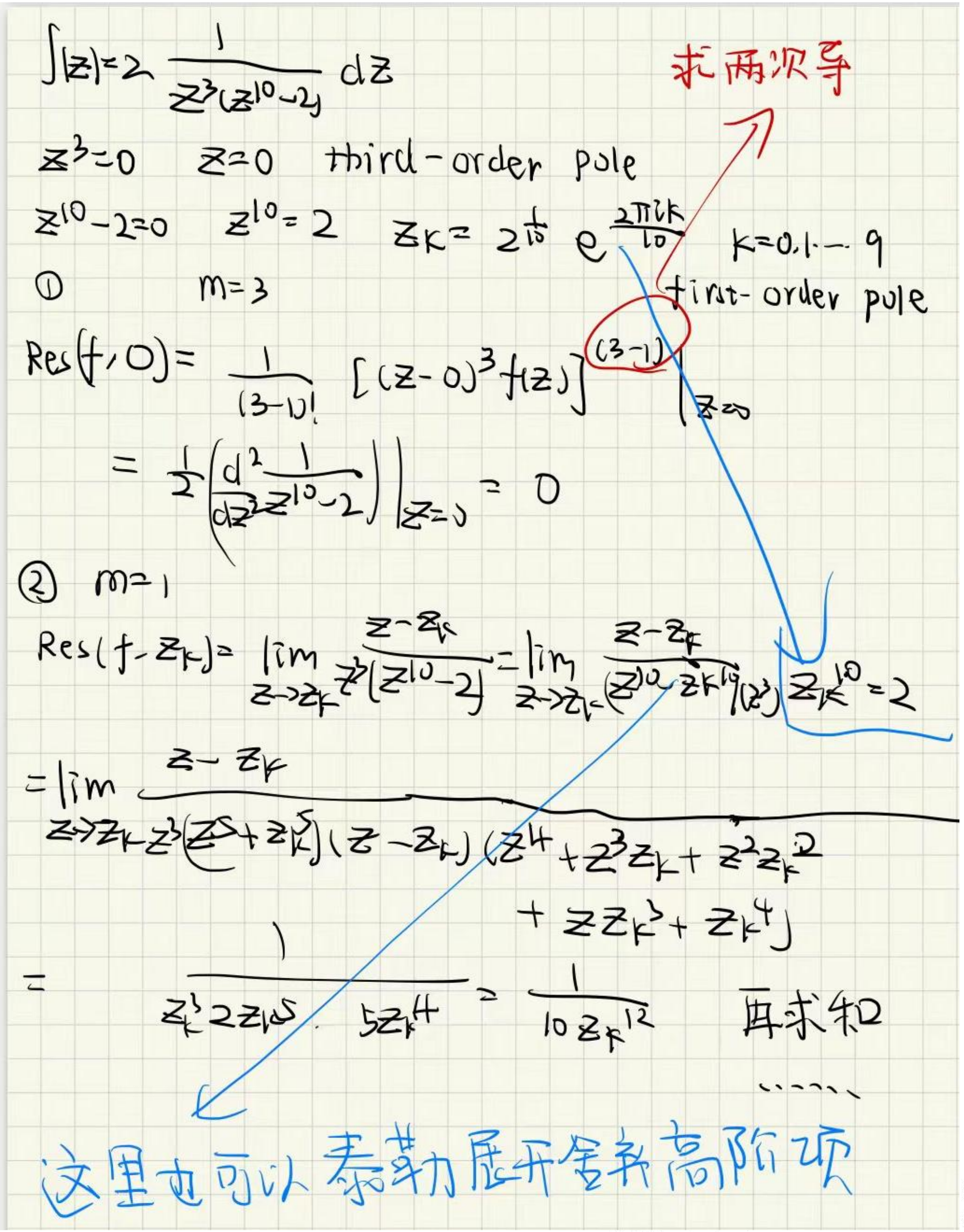
其中 $\omega_0 = 1$ 是所谓的“主单位根”。

性质和应用

1. **对称性**：单位根在复平面上等距分布，构成正 n 边形，且以原点为中心。
2. **乘法闭合**：单位根的乘积仍然是单位根。具体来说，$\omega_k \cdot \omega_m = \omega_{(k+m) \bmod n}$。
3. **乘方规律**：每一个单位根的 n 次方等于 1，即 $\omega_k^n = 1$。
4. **单位根的和**：所有 n 个单位根的和为零：

$$\sum_{k=0}^{n-1} \omega_k = 0$$

5. **常用单位根**：在 $n = 2, 3, 4, 6$ 的情况下，单位根有特殊的几何意义。例如：
   - 当 $n = 2$ 时，单位根是 1 和 -1。
   - 当 $n = 3$ 时，单位根是 1、ω、ω^2，其中 $\omega = e^{2\pi i/3}$。
   - 当 $n = 4$ 时，单位根是 1、i、-1、$-i$。



$\int_{|z|=2} \frac{1}{z^3(z^{10}-2)} dz$

求两次导

$z^3=0$ $z=0$ third-order pole

$z^{10}-2=0$ $z^{10}=2$ $z_k = 2^{\frac{1}{10}} e^{\frac{2\pi i k}{10}}$ $k=0,1\cdots 9$

first-order pole

① $m=3$

$$\mathrm{Res}(f,0) = \frac{1}{(3-1)!}\left[(z-0)^3 f(z)\right]^{(3-1)}\Big|_{z=0}$$

$$= \frac{1}{2}\left(\frac{d^2}{dz^2}\frac{1}{z^{10}-2}\right)\Big|_{z=0} = 0$$

② $m=1$

$$\mathrm{Res}(f,z_k) = \lim_{z\to z_k} \frac{z-z_k}{z^3(z^{10}-2)} = \lim_{z\to z_k} \frac{z-z_k}{(z^{10}-z_k^{10})(z^3)} \quad z_k^{10}=2$$

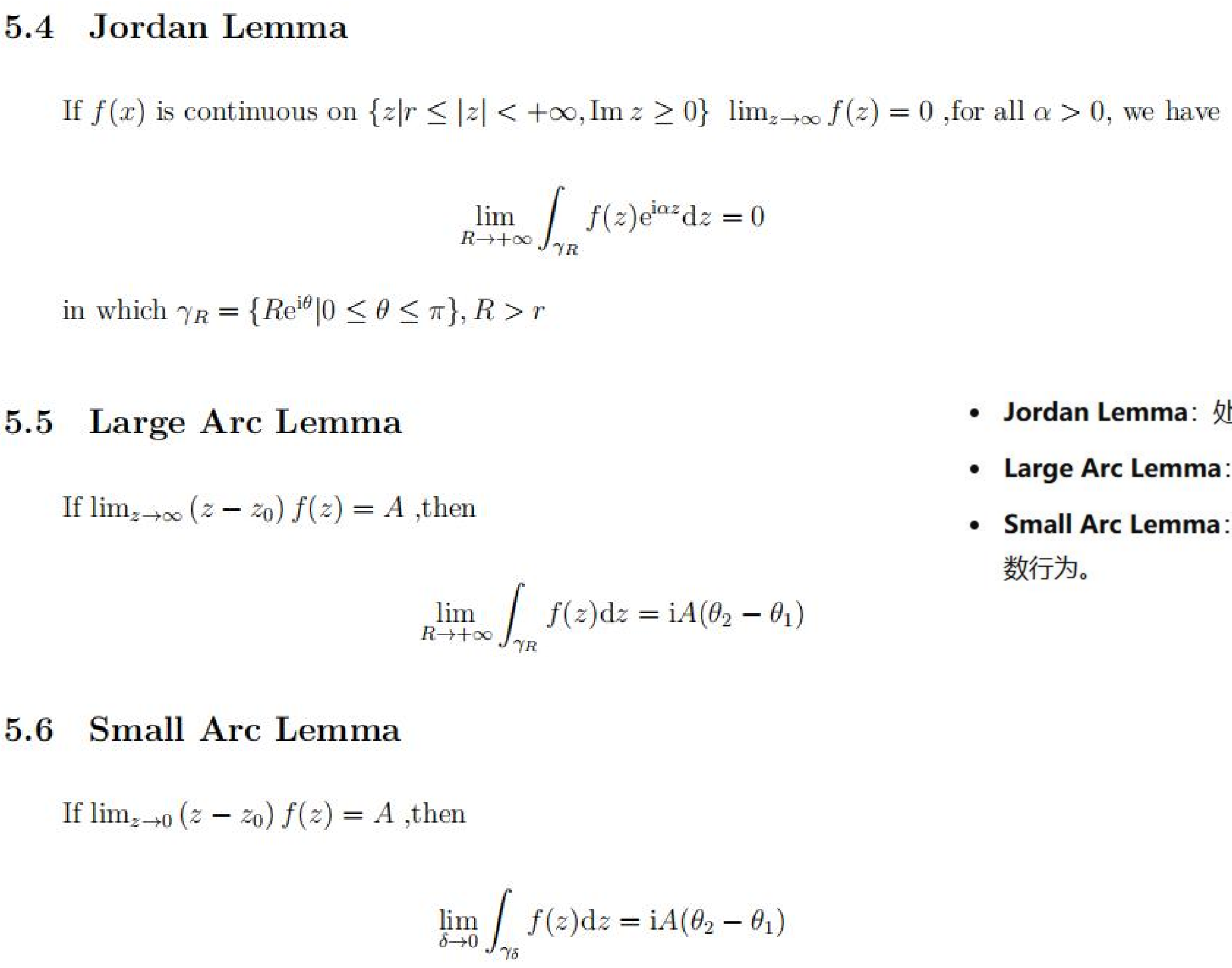
$$= \lim_{z\to z_k} \frac{z-z_k}{z^3(z^5+z_k^5)(z-z_k)(z^4+z^3z_k+z^2z_k^2+zz_k^3+z_k^4)}$$

$$= \frac{1}{z_k^3\, 2z_k^5 \cdot 5z_k^4} = \frac{1}{10z_k^{12}}$$ 再求和

……

这里也可以泰勒展开舍弃高阶项

There Lemmas (Very Important)



5.4 Jordan Lemma

If $f(x)$ is continuous on $\{z|r \leq |z| < +\infty, \operatorname{Im} z \geq 0\}$ $\lim_{z\to\infty} f(z) = 0$,for all $\alpha > 0$, we have

$$\lim_{R\to+\infty} \int_{\gamma_R} f(z)\mathrm{e}^{\mathrm{i}\alpha z}\mathrm{d}z = 0$$

in which $\gamma_R = \{R\mathrm{e}^{\mathrm{i}\theta}|0 \leq \theta \leq \pi\}, R > r$

5.5 Large Arc Lemma

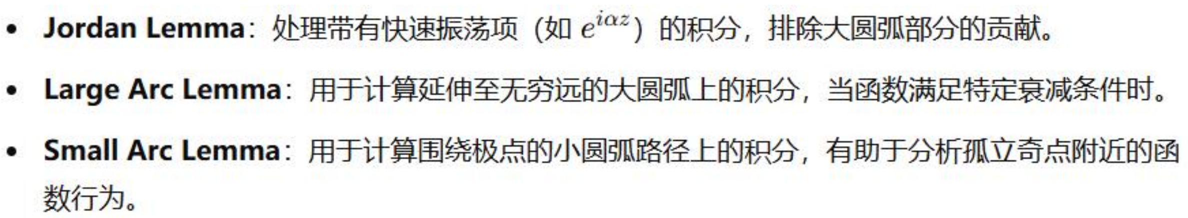
If $\lim_{z\to\infty} (z - z_0) f(z) = A$,then

$$\lim_{R\to+\infty} \int_{\gamma_R} f(z)\mathrm{d}z = \mathrm{i}A(\theta_2 - \theta_1)$$

5.6 Small Arc Lemma

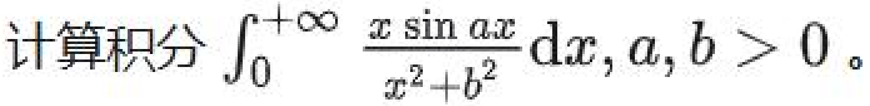
If $\lim_{z\to0} (z - z_0) f(z) = A$,then

$$\lim_{\delta\to0} \int_{\gamma_\delta} f(z)\mathrm{d}z = \mathrm{i}A(\theta_2 - \theta_1)$$

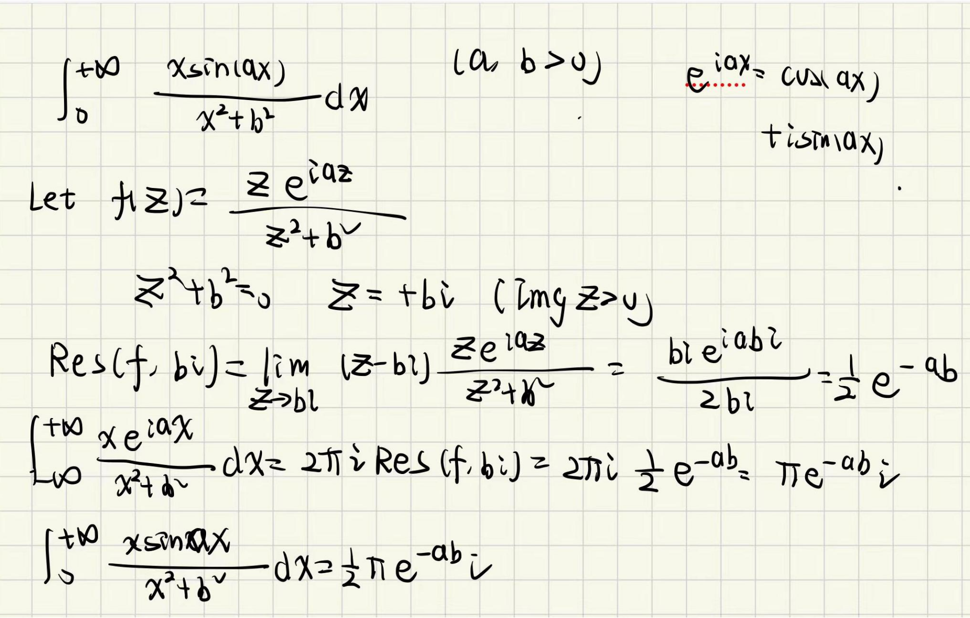


- **Jordan Lemma**：处理带有快速振荡项（如 $e^{i\alpha z}$）的积分，排除大圆弧部分的贡献。
- **Large Arc Lemma**：用于计算延伸至无穷远的大圆弧上的积分，当函数满足特定衰减条件时。
- **Small Arc Lemma**：用于计算围绕极点的小圆弧路径上的积分，有助于分析孤立奇点附近的函数行为。

Jordan lemma Q3



计算积分 $\int_0^{+\infty}\frac{x\sin ax}{x^2+b^2}\mathrm{d}x, a,b>0$ 。



$\int_0^{+\infty}\frac{x\sin(ax)}{x^2+b^2}dx$ $(a, b>0)$ $e^{iax}=\cos(ax)+i\sin(ax)$

Let $f(z)=\frac{ze^{iaz}}{z^2+b^2}$

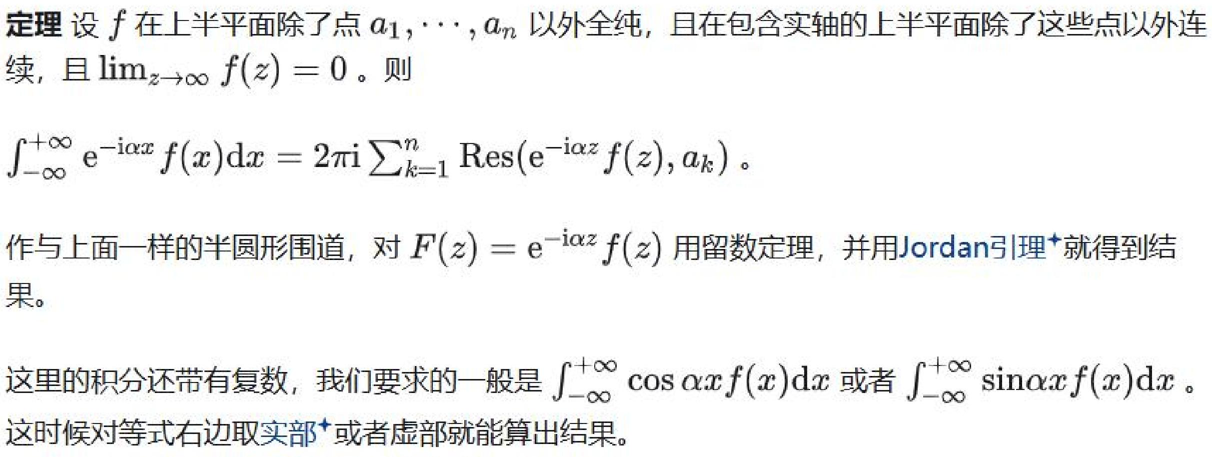
$z^2+b^2=0$ $z=+bi$ $(\operatorname{Img} z>0)$

$\operatorname{Res}(f, bi)=\lim\limits_{z\to bi}(z-bi)\frac{ze^{iaz}}{z^2+b^2}=\frac{bie^{iabi}}{2bi}=\frac{1}{2}e^{-ab}$

$\int_{-\infty}^{+\infty}\frac{xe^{iax}}{x^2+b^2}dx=2\pi i\operatorname{Res}(f, bi)=2\pi i\frac{1}{2}e^{-ab}=\pi e^{-ab}i$

$\int_0^{+\infty}\frac{x\sin ax}{x^2+b^2}dx=\frac{1}{2}\pi e^{-ab}i$

TA wld didn't include this kind of problems and I have no idea whether it will be tested.

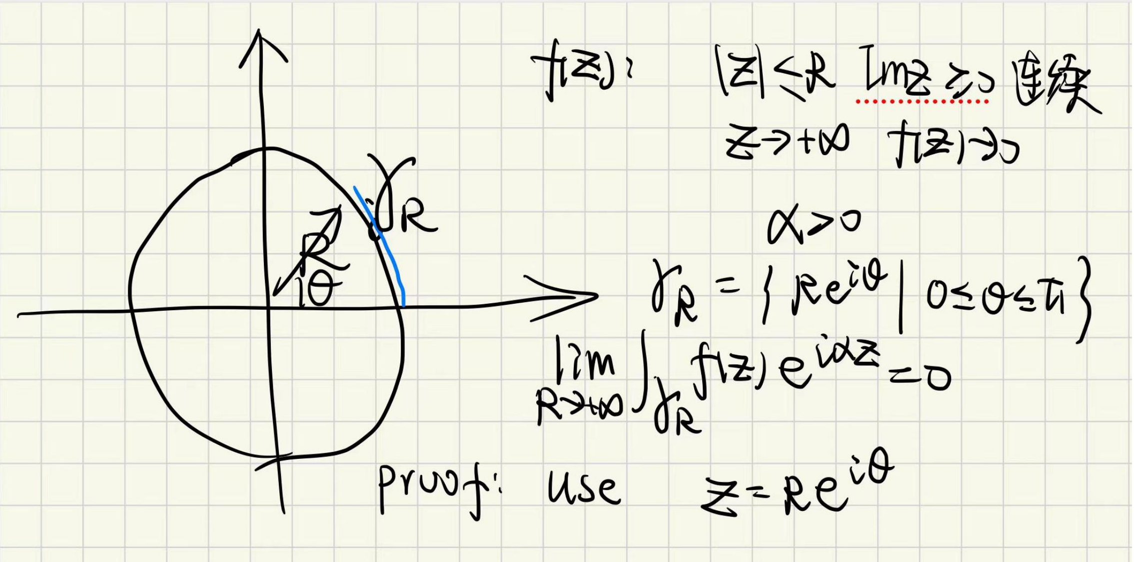


定理 设 f 在上半平面除了点 $a_1,\cdots,a_n$ 以外全纯，且在包含实轴的上半平面除了这些点以外连续，且 $\lim_{z\to\infty}f(z)=0$ 。则

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\alpha x}f(x)\mathrm{d}x=2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname{Res}(\mathrm{e}^{-\mathrm{i}\alpha z}f(z),a_k)\text{ 。}$$

作与上面一样的半圆形围道，对 $F(z)=\mathrm{e}^{-\mathrm{i}\alpha z}f(z)$ 用留数定理，并用Jordan引理⁺就得到结果。

这里的积分还带有复数，我们要求的一般是 $\int_{-\infty}^{+\infty}\cos\alpha xf(x)\mathrm{d}x$ 或者 $\int_{-\infty}^{+\infty}\sin\alpha xf(x)\mathrm{d}x$ 。这时候对等式右边取实部⁺或者虚部就能算出结果。



$f(z)$: $|z|\le R$ $\operatorname{Im}z\ge0$ 连续

$z\to+\infty$ $f(z)\to0$

$\alpha>0$

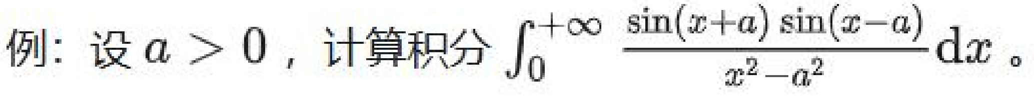
$\gamma_R=\{Re^{i\theta}\mid 0\le\theta\le\pi\}$

$\lim\limits_{R\to+\infty}\int_{\gamma_R}f(z)e^{i\alpha z}=0$

proof: use $z=Re^{i\theta}$

Small Arc Lemma Q4

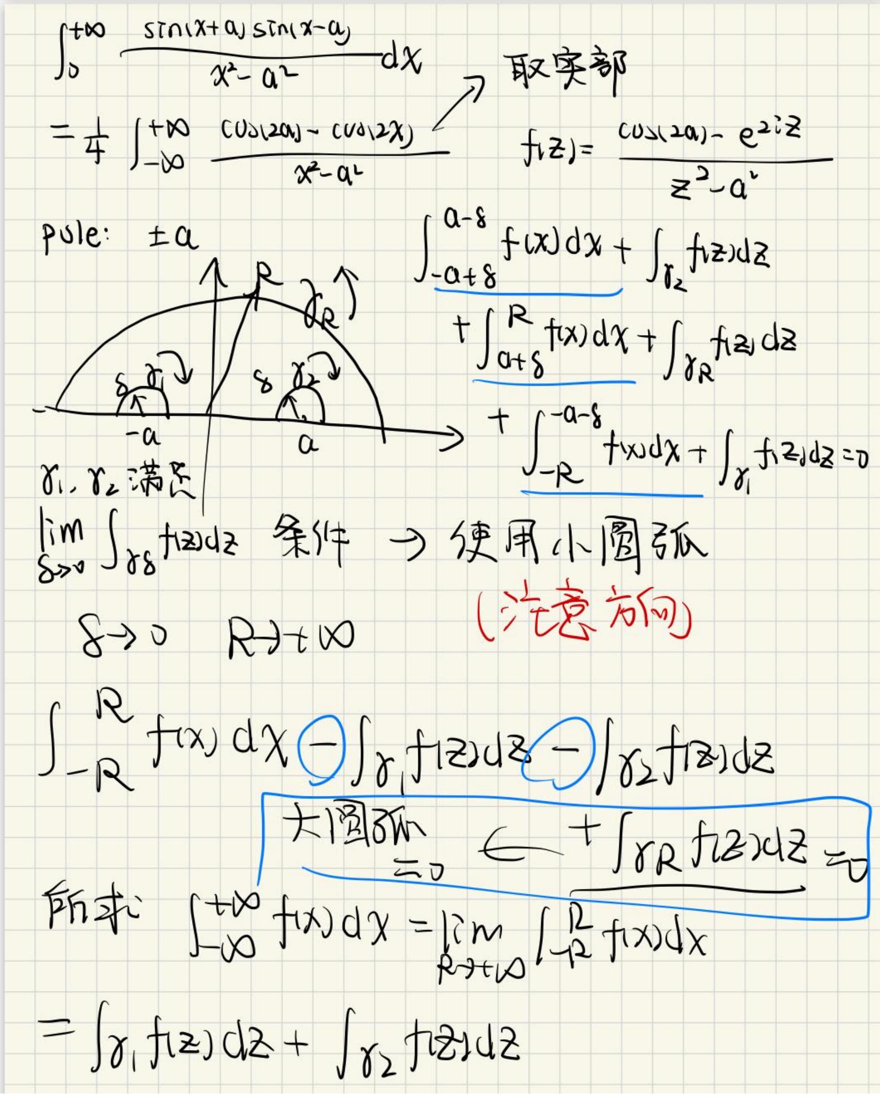
引理的一般用法是，在奇点周围挖一块扇形区域，对这一块用小圆弧引理。但是我们一般不考察复杂的奇点，这里我们的奇点都视作之前提到的pole就行



例：设 $a>0$，计算积分 $\int_0^{+\infty}\frac{\sin(x+a)\sin(x-a)}{x^2-a^2}\mathrm{d}x$。

本题大概也不会考，更多考察的是分母里为虚根的情况
本题只供理解画一个大圈圈的方法，不要求掌握

Large Arc Lemma



$$\int_0^{+\infty}\frac{\sin(x+a)\sin(x-a)}{x^2-a^2}dx$$

$$=\frac{1}{4}\int_{-\infty}^{+\infty}\frac{\cos(2a)-\cos(2x)}{x^2-a^2}\nearrow \text{取实部}$$

$$f(z)=\frac{\cos(2a)-e^{2iz}}{z^2-a^2}$$

pole: $\pm a$

$$\underline{\int_{-a+\delta}^{a-\delta}f(x)dx}+\int_{\gamma_2}f(z)dz$$

$$+\underline{\int_{a+\delta}^{R}f(x)dx}+\int_{\gamma_R}f(z)dz$$

$$+\underline{\int_{-R}^{-a-\delta}f(x)dx}+\int_{\gamma_1}f(z)dz=0$$

γ_1, γ_2 满足

$\lim\limits_{\delta\to 0}\int_{\gamma_\delta}f(z)dz$ 条件 → 使用小圆弧

(注意方向)

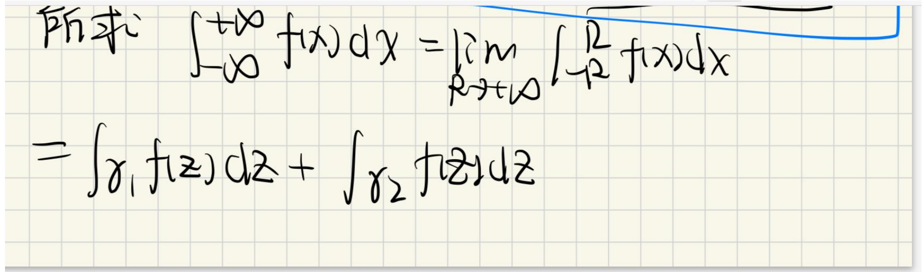
$\delta\to 0$ $R\to+\infty$

$$\int_{-R}^{R}f(x)\,dx\boxed{-}\int_{\gamma_1}f(z)dz\boxed{-}\int_{\gamma_2}f(z)dz$$

$$\boxed{\underset{=0}{\text{大圆弧}}\longleftarrow+\underline{\int_{\gamma_R}f(z)dz}=0}$$

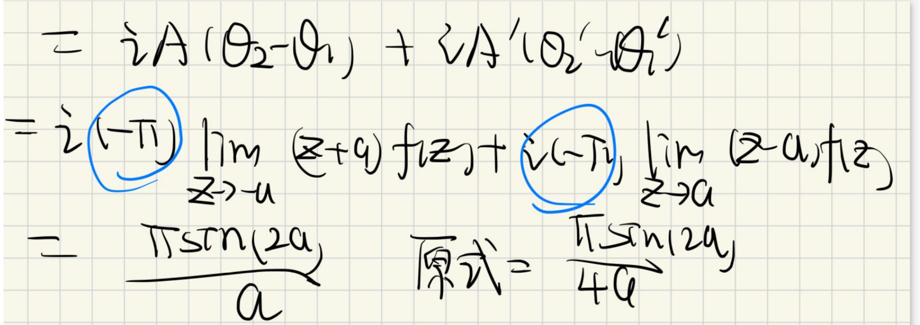
所求 $\int_{-\infty}^{+\infty}f(x)\,dx=\lim\limits_{R\to+\infty}\int_{-R}^{R}f(x)dx$

$$=\int_{\gamma_1}f(z)dz+\int_{\gamma_2}f(z)dz$$



所求 $\int_{-\infty}^{+\infty}f(x)\,dx=\lim\limits_{R\to+\infty}\int_{-R}^{R}f(x)dx$

$$=\int_{\gamma_1}f(z)dz+\int_{\gamma_2}f(z)dz$$



$$=iA(\theta_2-\theta_1)+iA'(\theta_2'-\theta_1')$$

$$=i\boxed{(-\pi)}\lim_{z\to -a}(z+a)f(z)+\boxed{i(-\pi)}\lim_{z\to a}(z-a)f(z)$$

$$=\frac{\pi\sin(2a)}{a}\qquad \text{原式}=\frac{\pi\sin(2a)}{4a}$$

Complex questions but very traditional
You may encounter silmilar questions in exam
But the question in exam may be in simpler form

一般以横轴从左往右为正，圆逆时针旋转为正

A series of questions (from TA wld’s RC)

He said these four types of questions are enough for exam (maybe)
I still have no idea and if I don’t use his examples, I may find too diffifcult questions, If you cannot understand the content before. These four questions may be enough most time.

Form of $\int_0^{2\pi} f(cosx, sinx)dx$

Form of $\int_0^{+\infty} f(x)\cos mx\mathrm{d}x$

Form of $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$

Form of real first order pole

Q5

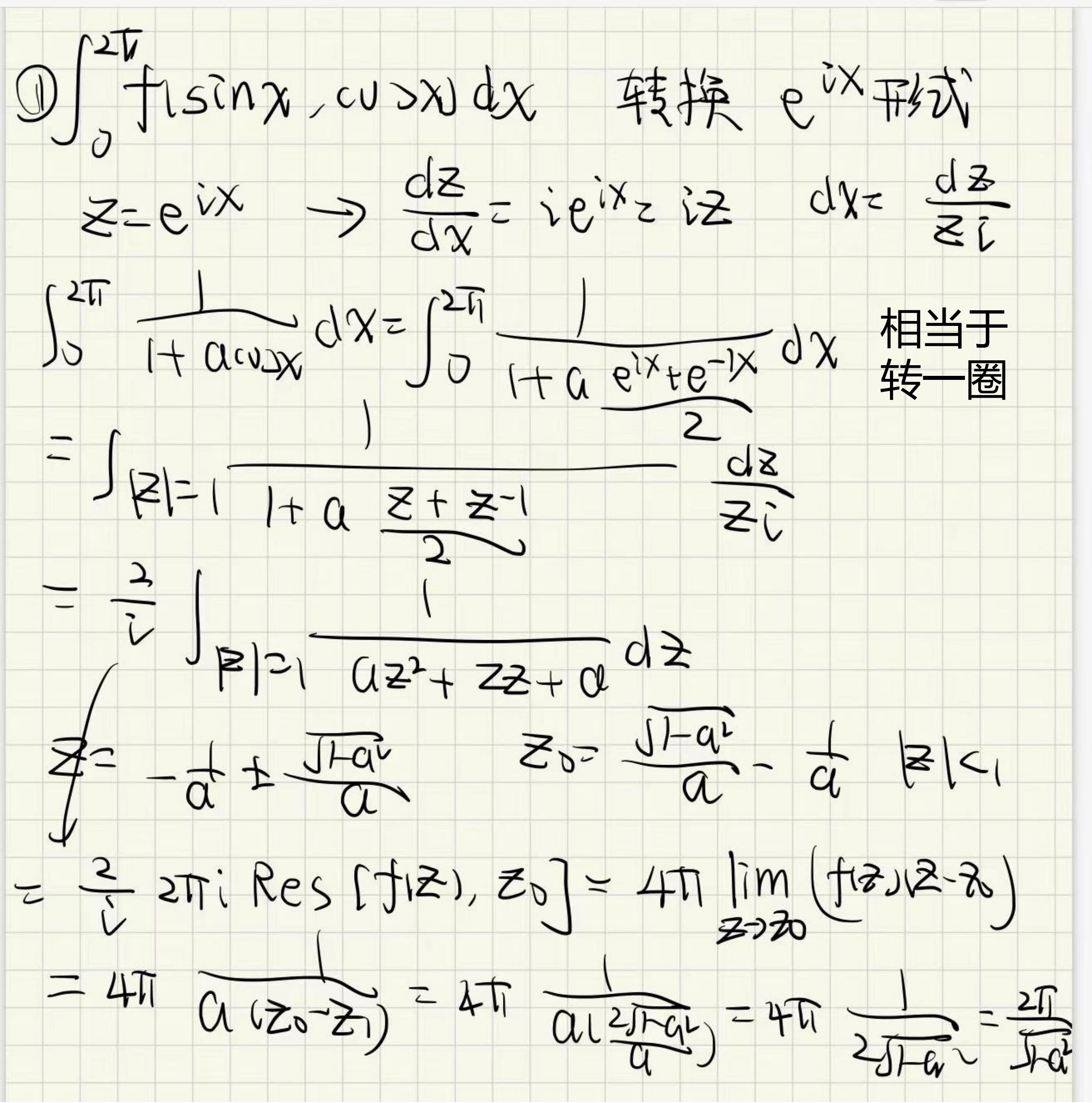
5.7 Form of $\int_0^{2\pi} f(cosx, sinx)dx$

Evaluate

$$\int_0^{2\pi} \frac{1}{1 + acosx} dx$$

In this kind of form, f is continuous function of $sinx$ and $cosx$.

Remember !!!



① $\int_0^{2\pi} f(\sin x, \cos x)\,dx$ 转换 e^{ix}形式

$z = e^{ix} \rightarrow \frac{dz}{dx} = ie^{ix} = iz \quad dx = \frac{dz}{zi}$

$\int_0^{2\pi} \frac{1}{1+a\cos x} dx = \int_0^{2\pi} \frac{1}{1+a\frac{e^{ix}+e^{-ix}}{2}} dx$ 相当于转一圈

$= \int_{|z|=1} \frac{1}{1+a\frac{z+z^{-1}}{2}} \frac{dz}{zi}$

$= \frac{2}{i} \int_{|z|=1} \frac{1}{az^2+2z+a} dz$

$z = -\frac{1}{a} \pm \frac{\sqrt{1-a^2}}{a} \quad z_0 = \frac{\sqrt{1-a^2}}{a} - \frac{1}{a} \quad |z|<1$

$= \frac{2}{i} 2\pi i \, \mathrm{Res}[f(z), z_0] = 4\pi \lim_{z \to z_0} (f(z)(z-z_0))$

$= 4\pi \frac{1}{a(z_0-z_1)} = 4\pi \frac{1}{a(\frac{2\sqrt{1-a^2}}{a})} = 4\pi \frac{1}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$

Q6

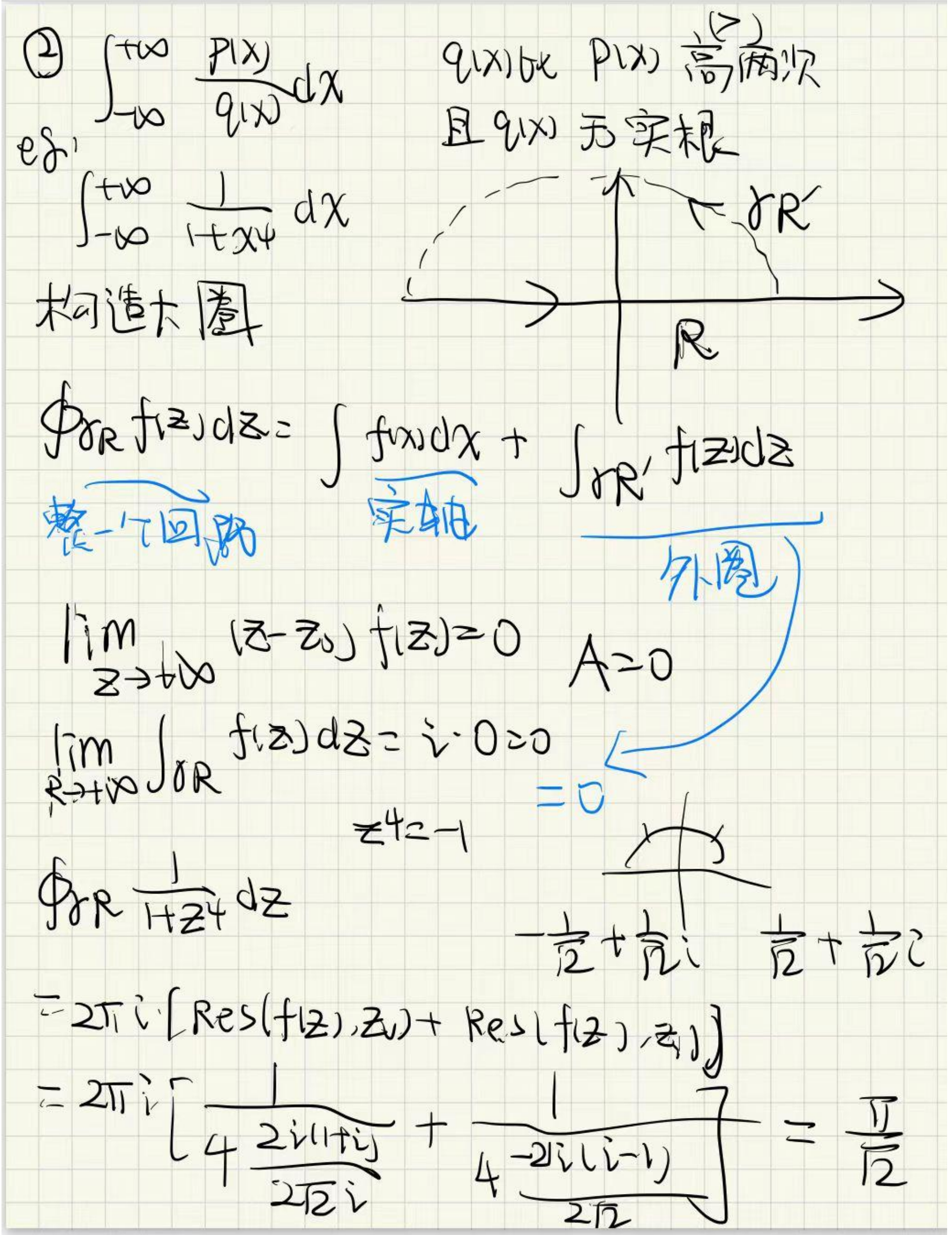
5.8 Form of $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx$

Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}$$

In this case,$q(x)$ always doesn't admit real roots(actually we can deal with the case with real root, but it is too hard for you.) And, $q(x)$ is always "higher" than $p(x)$ by at least 2 orders.

Remember !!!



② $\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)} dx$ q(x) 比 p(x) 高(≥)两次

且 q(x) 无实根

eg: $\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx$

构造大圈

$\oint_{\gamma R} f(z) dz = \underbrace{\int f(x)dx}_{实轴} + \underbrace{\int_{\gamma R'} f(z)dz}_{外圈}$

整一个回路

$\lim_{z\to+\infty} (z-z_0) f(z) = 0$ A=0

$\lim_{R\to+\infty} \int_{\gamma R} f(z) dz = i \cdot 0 = 0$

=0

$z^4=-1$

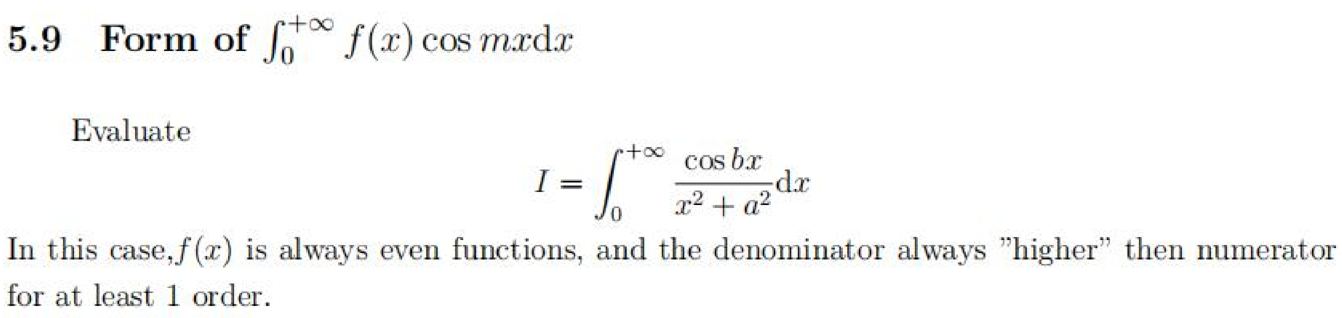
$-\frac{1}{\sqrt2}+\frac{1}{\sqrt2}i$ $\frac{1}{\sqrt2}+\frac{1}{\sqrt2}i$

$\oint_{\gamma R} \frac{1}{1+z^4} dz$

$= 2\pi i [\text{Res}(f(z), z_0) + \text{Res}(f(z), z_1)]$

$= 2\pi i \left[\frac{1}{4\frac{2i(1+i)}{2\sqrt2 i}} + \frac{1}{4\frac{-2i(i-1)}{2\sqrt2}} \right] = \frac{\pi}{\sqrt2}$

Q7



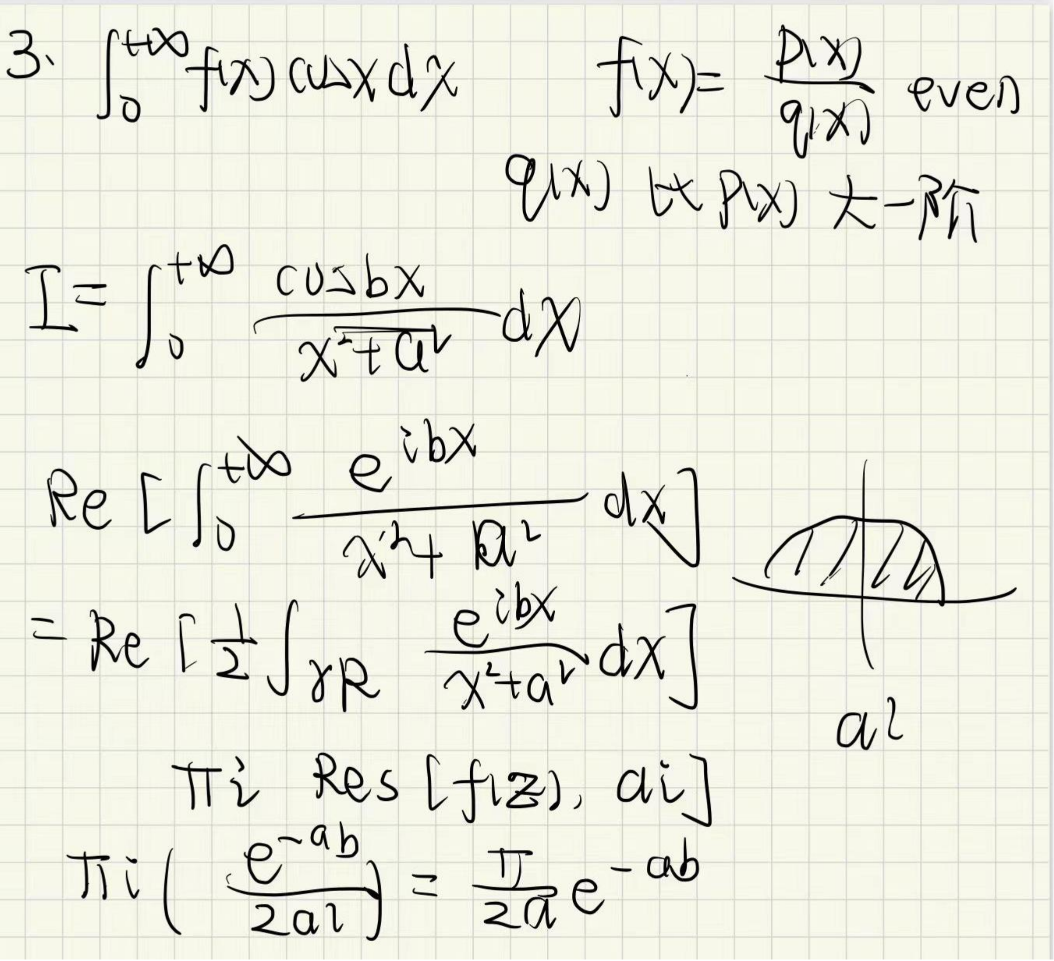
5.9 Form of $\int_0^{+\infty} f(x)\cos mx\,\mathrm{d}x$

Evaluate

$$I = \int_0^{+\infty} \frac{\cos bx}{x^2+a^2}\mathrm{d}x$$

In this case, $f(x)$ is always even functions, and the denominator always "higher" then numerator for at least 1 order.

Remember !!!



3、 $\int_0^{+\infty} f(x)\cos x\,dx$ $\quad f(x)=\frac{p(x)}{q(x)}$ even

$q(x)$ 比 $p(x)$ 大一阶

$$I=\int_0^{+\infty}\frac{\cos bx}{x^2+a^2}dx$$

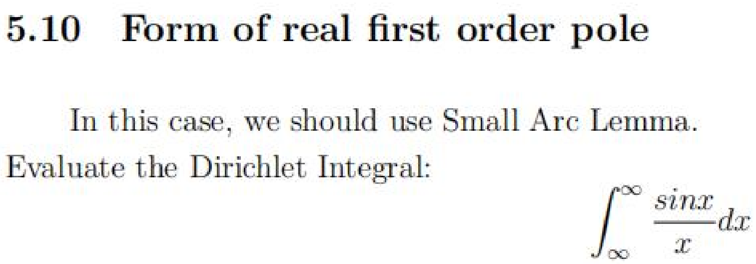
$$\mathrm{Re}\left[\int_0^{+\infty}\frac{e^{ibx}}{x^2+a^2}dx\right]$$

$$=\mathrm{Re}\left[\frac{1}{2}\int_{\gamma R}\frac{e^{ibx}}{x^2+a^2}dx\right]$$

$$\pi i\ \mathrm{Res}[f(z), ai]$$

$$\pi i\left(\frac{e^{-ab}}{2ai}\right)=\frac{\pi}{2a}e^{-ab}$$

Q8

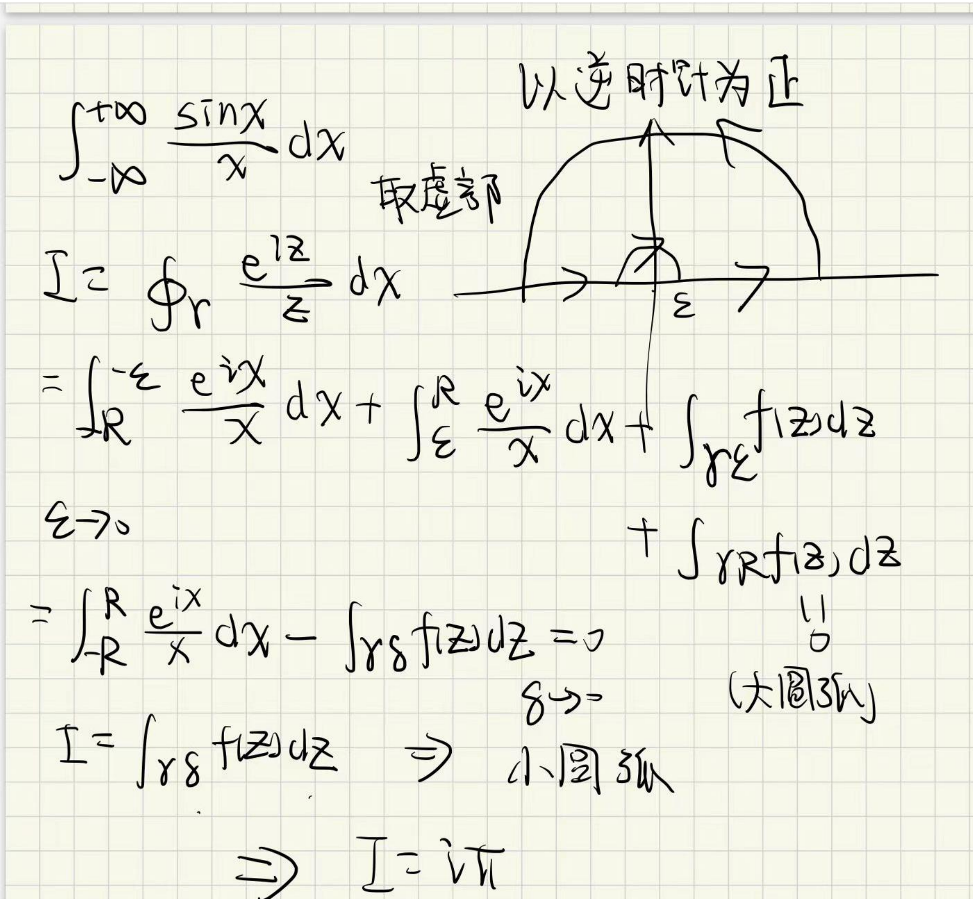


5.10 Form of real first order pole

In this case, we should use Small Arc Lemma. Evaluate the Dirichlet Integral:

$$\int_{\infty}^{\infty} \frac{sinx}{x} dx$$

Remember !!!



$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx$$

取虚部

$$I = \oint_{r} \frac{e^{iz}}{z} dx$$

$$= \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x} dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x} dx + \int_{\gamma_\varepsilon} f(z) dz$$

$$+ \int_{\gamma_R} f(z) dz$$

$$= 0$$

(大圆弧)

$\varepsilon \to 0$

$$= \int_{-R}^{R} \frac{e^{ix}}{x} dx - \int_{\gamma_\delta} f(z) dz = 0$$

$\delta \to 0$

$$I = \int_{\gamma_\delta} f(z) dz \Rightarrow$$ 小圆弧

$$\Rightarrow I = i\pi$$

Thank You !!!

From The Elder Scrolls V : Skyrim Special Edition