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## VV256 RC3

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# Contents

- Laplace Transform
- Solution of Initial Value Problem
- Differential Equations with Discontinuous Forcing Functions
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# Integral Transform

A very useful tool for solving linear differential equations.

Basic form: $F(s) = \int_{\alpha}^{\beta} K(s,t)f(t)dt$, where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration $\alpha$ and $\beta$ are also given. It's possible that $\alpha = -\infty$, or $\beta = -\infty$ or both.

Objective: Transforming the function $f$ into another function $F$.

Classification: Laplace Transform, Fourier Transform, Z Transform . . .

In this course, we will meet Laplace Transform and Fourier Transfomr

# Laplace Transform

Definition: Let $f(t)$ be given for $t \geq 0$, and suppose that $f$ satisfies certain conditions (will be stated later). Then the Laplace transform of $f$, which we will denote by $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$.

Why we use Laplace Transform ?

拉普拉斯变换法主要是借助拉普拉斯变换把常系数线性微分方程（组）转换成复变数s的代数方程组，通过拉普拉斯表和相关性质快速解决常见简单的微分方程，但是如果仅适用于微分方程的右端函数的拉普拉斯表里的原函数

VV256 just tests a little part about Laplace Transform and if you want to study more, you can just search some articles in CSDN or zhihu or wiki or Google. There are a lot of aritcles.

Indeed, most parts of Laplace Transform will be provided as a referrence table in the exam and you don’t need to remember all equation.
But you should be famaliar with how to get those Laplace Transformation and there exists some useful techiniques in changing the complex equations into simple and famaliar forms.

# Laplace Table

| | $f(t)$ | $F(s)$ |
|---|---|---|
| 1. | $1$ | $\frac{1}{s}, \quad s > 0$ |
| 2. | $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| 3. | $t^n, \quad n =$ positive integer | $\frac{n!}{s^{n+1}}, \quad s > 0$ |
| 4. | $t^p, \quad p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$ |
| 5. | $\sin at$ | $\frac{a}{s^2+a^2}, \quad s > 0$ |
| 6. | $\cos at$ | $\frac{s}{s^2+a^2}, \quad s > 0$ |
| 7. | $\sinh at$ | $\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$ |
| 8. | $\cosh at$ | $\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$ |
| 9. | $e^{at} \sin bt$ | $\frac{b}{(s-a)^2+b^2}, \quad s > a$ |
| 10. | $e^{at} \cos bt$ | $\frac{s-a}{(s-a)^2+b^2}, \quad s > a$ |
| 11. | $t^n e^{at}, \quad n =$ positive integer | $\frac{n!}{(s-a)^{n+1}}, \quad s > a$ |
| 12. | $u_c(t)$ | $\frac{e^{-cs}}{s}, \quad s > 0$ |
| 13. | $u_c(t) f(t-c)$ | $e^{-cs} F(s)$ |
| 14. | $e^{ct} f(t)$ | $F(s-c)$ |
| 15. | $f(ct)$ | $\frac{1}{c} F\left(\frac{s}{c}\right), \quad c > 0$ |
| 16. | $\int_0^t f(t-\tau) g(\tau)\, d\tau$ | $F(s)G(s)$ |
| 17. | $\delta(t-c)$ | $e^{-cs}$ |
| 18. | $f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$ |
| 19. | $(-t)^n f(t)$ | $F^{(n)}(s)$ |

# Q1:

Determine the Laplace Transform of $f(t)$

1) $f(t)=1$ $\quad L\{1\}=\int_0^{+\infty} e^{-st}\,dt=$

$-\lim_{A\to+\infty} \frac{e^{-st}}{s}\Big|_0^A = \frac{1}{s} \quad s>0$

2) $f(t)=at$ $\quad L\{at\}=\int_0^{+\infty} e^{-(s-a)t} = \frac{1}{s-a} \quad s>a$

3) $f(t)=\cos t$ $\quad \cos t=\frac{e^{it}+e^{-it}}{2}$

$L\{f(t)\}=\int_0^{+\infty} \frac{e^{it}+e^{-it}}{2} e^{-st}\,dt$

$=\frac{1}{2}\int_0^{+\infty} e^{-(s-i)t}+e^{-(s+i)t} = \frac{1}{2}\left[\frac{1}{s-i}+\frac{1}{s+i}\right]$

$=\frac{s}{s^2+1}$

Something you need to recall:

$$8.\ e^{i\theta}=\cos\theta+i\sin\theta,\ e^{-i\theta}=\cos\theta-i\sin\theta$$

$$9.\ \cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2},\ \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

# Linearity of Laplace Transform

Let us suppose that $f_1$ and $f_2$ are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for $s$ greater than the maximum of $a_1$ and $a_2$,

$L\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]dt =$
$c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty e^{-st} f_2(t)dt.$

Hence, Ł$\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}.$

This indicates that Laplace transform is a linear operator. Meanwhile, the sum can be extended to an arbitrary number of terms.

# Differentiation Theorem of the Laplace Transform

Suppose that $f$ is continuous and $f'$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants $K$, $a$ and $M$ such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then $L\{f'(t)\}$ exists for $s > a$, and moreover,

$L\{f'(t)\} = sL\{f(t)\} - f(0).$

# Solution of Initial Value Problems

$L\{f'(t)\} = sL\{f(t)\} - f(0).$

$L\{f''(t)\} = sL\{f'(t)\} - f'(0) = s[sL\{f(t)\} - f(0)] - f'(0) =$
$s^2L\{f(t)\} - sf(0) - f'(0).$

What if $n$ gets larger...?

$L\{f^{(n)}\} = s^nL\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$

# Q2

Use the Laplace transform to solve the given initial value problem.

$y'' - y' - 6y = 0; y(0) = 1, y'(0) = -1.$

Use the Laplace transform to solve the given initial value problem.

$y'' - y' - 6y = 0; y(0) = 1, y'(0) = -1.$

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) = 0$$

$$[s^2Y(s) - s + 1] - [sY(s) - 1] - 6Y(s) = 0$$

$$(s^2 - s - 6)Y(s) = s - 2$$

$$Y(s) = \frac{s-2}{(s-3)(s+2)} = \frac{1/5}{s-3} + \frac{4/5}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

# Q3

$x'' + 2x' + x = e^{-t}$ $\quad x(1) = x'(1) = 0$

Let $\tau = t-1$

$x'' + 2x' + x = e^{-(\tau+1)}$ $\quad x(0) = x'(0) = 0$

$s^2 X(s) + 2sX(s) + X(s) = \frac{1}{s+1} \cdot \frac{1}{e}$

$X(s) = \frac{1}{(s+1)^3} \frac{1}{e}$ 查表

$x(\tau) = \frac{1}{2} \tau^2 e^{-\tau-1}$ $\quad x(t) = \frac{1}{2}(t-1)^2 e^{-t}$

# Q5

Use linearity to determine the Inverse Laplace Transform of the following $F(s)$:

(1) $F(s) = \frac{3}{s^2+4}$; $f(t) = \frac{3}{2}\sin 2t$

(2) $F(s) = \frac{2}{s^2+3s-4}$; $f(t) = \frac{2}{5}e^{t} - \frac{2}{5}e^{-4t}$

(3) $F(s) = \frac{2s+2}{s^2+2s+5}$; $f(t) = 2e^{-t}\cos 2t$

(4) $F(s) = \frac{2s+1}{s^2-2s+2}$; $f(t) = 2e^{t}\cos t + 3e^{t}\sin t$

(5) $F(s) = \frac{8s^2-4s+12}{s(s^2+4)}$. $f(t) = 3 + 5\cos 2t - 2\sin 2t$

Try yourself !!!

# Step Function

To deal effectively with functions having jump discontinuities, it's helpful to introduce a function known as the unit step function or Heaviside function.

Form: $u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$.

Example: For the function $f(t) = \begin{cases} 2, & 0 \leq t < 4 \\ 5, & 4 \leq t < 7 \\ -1, & 7 \leq t < 9 \\ 1, & t \geq 9 \end{cases}$, it can be written as the form of step function: $f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$.

# Time-Frequency Interchange Theorem

Theorem1: If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then
$L\{u_c(t)f(t-c)\} = e^{-cs}L\{f(t)\} = e^{-cs}F(s), s \geq a.$

Conversely, if $f(t) = L^{-1}\{F(s)\}$, then
$u_c(t)f(t-c) = L^{-1}\{e^{-cs}F(s)\}$

Theorem2: If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if $c$ is a constant, then $Le^{ct}f(t) = F(s-c), s > a + c.$

Conversely, if $f(t) = L^{-1}\{F(s)\}$, then $e^{ct}f(t) = L^{-1}\{F(s-c)\}$.

# Q6

$f(t) = H(t-1) + 2H(t-3) - 6H(t-4)$

$F(s) = \dfrac{e^{-s} + 2e^{-3s} - 6e^{-4s}}{s}$

(1) Find the Laplace transform of the given function:

$f(t) = u_1(t) + 2u_3(t) - 6u_4(t);$

(2) Find the inverse Laplace transform of the given functions:

$F(s) = \frac{3!}{(s-2)^4};$ $f(t) = t^3 e^{2t}$

$F(s) = \frac{e^{-2s}}{s^2+s-2};$ $f(t) = \frac{1}{3}[e^{t-2} - e^{-2(t-2)}]u_2(t)$

$F(s) = \frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}.$ $f(t) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$

# Some special techniques in partial-fraction expansion

If you meet complex F(s) like $\frac{1}{as^2+bs+c}$ or higher order $\frac{1}{as^n+bs^m+\cdots}$ or $\frac{ks}{as^2+bs+c}$

you need to simplify

① $$F(s)=\frac{1}{s^3+3s^2+2s}=\frac{1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

$$A=sF(s)\big|_{s=0}=\frac{1}{1\times 2}=\frac{1}{2}$$

$$B=(s+1)F(s)\big|_{s=-1}=\frac{1}{-1\times 1}=-1$$

$$C=(s+2)F(s)\big|_{s=-2}=\frac{1}{-2\times(-1)}=\frac{1}{2}$$

② $$F(s)=\frac{s+4}{s^2+3s+2}=\frac{A}{s+1}+\frac{B}{s+2}$$

$$A=(s+1)F(s)\big|_{s=-1}=\frac{s+4}{s+2}\Big|_{s=-1}=\frac{3}{1}=3$$

$$B=(s+2)F(s)\big|_{s=-2}=\frac{s+4}{s+1}\Big|_{s=-2}=\frac{2}{-1}=-2$$

$$F(s)=\frac{3}{s+1}-\frac{2}{s+2}$$

If you are still interested in these parts, I upload the pfe part (from VE216) in canvas.

# Important Properties

Suppose that $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$.

(a) If $c$ is a positive constant, then $L\{f(ct)\} = \frac{1}{c}F(\frac{s}{c}), s > ca$.

(b) If $k$ is a positive constant, then $L^{-1}\{F(ks)\} = \frac{1}{k}f(\frac{t}{k})$.

(c) If $a$ and $b$ are constants with $a > 0$, then

$L^{-1}\{F(as+b)\} = \frac{1}{a}e^{\frac{-bt}{a}}f(\frac{t}{a})$.

Solve the following differential equations with discontinuous forcing functions:

(1) $y'' + y = f(t); y(0) = 0, y'(0) = 1; f(t) = \begin{cases} 1, & 0 \leq t < 3\pi \\ 0, & 3\pi \leq t < \infty \end{cases}$;

$y(t) = 1 + \sin t - \cos t - (1+\cos t)u_{3\pi}(t)$

(2) $y'' + y = g(t); y(0) = 0, y'(0) = 1; g(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < 6 \\ 3, & 6 \leq t < \infty \end{cases}$.

$y(t) = \frac{1}{2}(t + \sin t) - \frac{1}{2}[(t-6) - \sin(t-6)]u_6(t)$

# The Convolution Integral

Definition: If $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$ both exist for $s > a \geq 0$, then $H(s) = F(s)G(s) = L\{h(t)\}, s > a$,
where $h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$
And the function $h$ is known as the convolution of $f$ and $g$; the integrals are called convolution integrals.

# Important Properties

Commutative law: $f * g = g * f$;

Distributive law: $f * (g_1 + g_2) = f * g_1 + f * g_2$;

Associative law: $(f * g) * h = f * (g * h)$;

Null identity: $f * 0 = 0 * f = 0$.

# Q7

$h(t)=\int_0^t \tau \sin a(t-\tau)d\tau$

(1) Find the inverse transform of $H(s) = \frac{a}{s^2(s^2+a^2)}$;

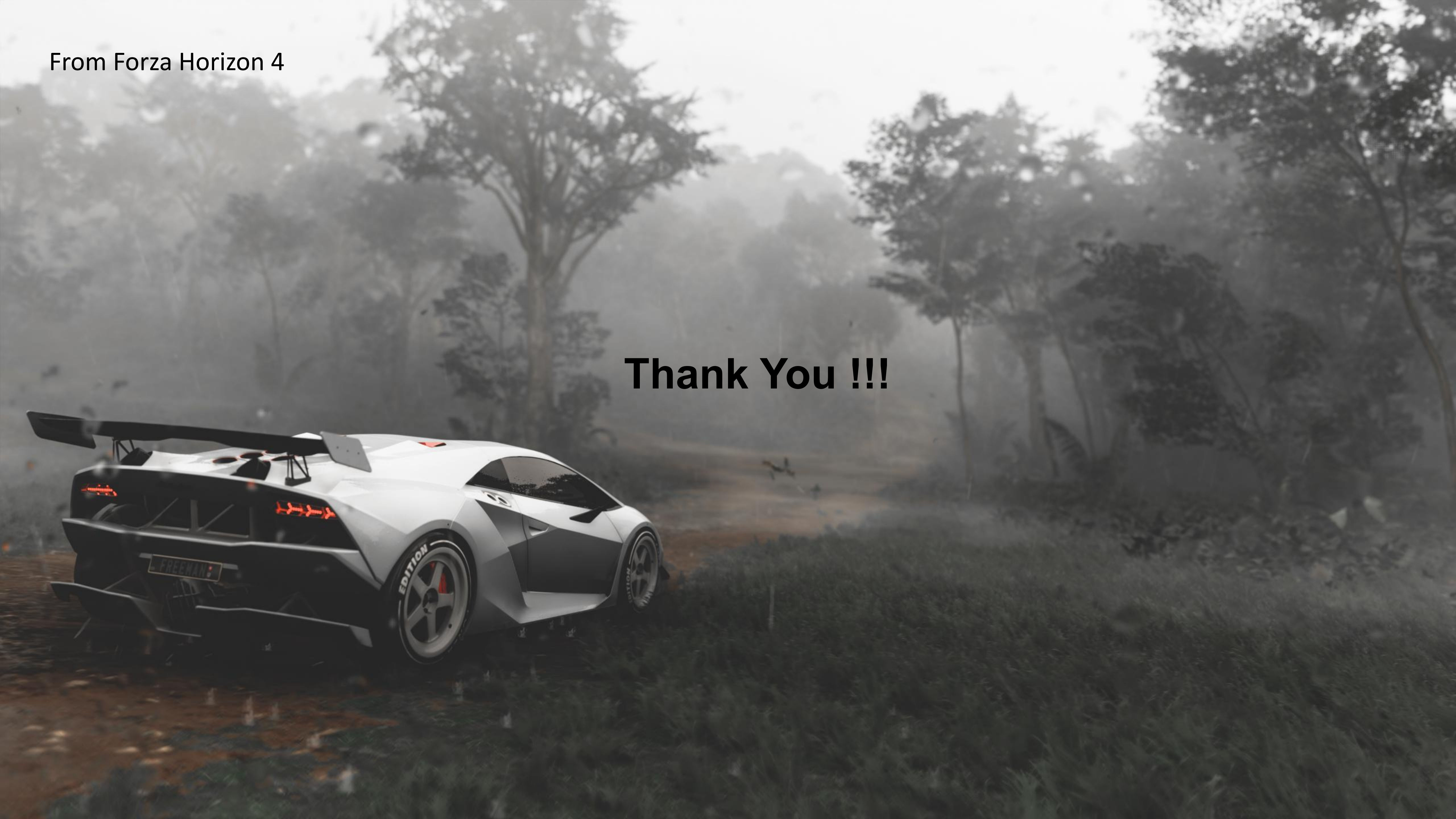
(2) Find the Laplace transform of $f(t) = \int_0^t (t-\tau)^2 cos 2\tau d\tau$; $\frac{2}{s^2(s^2+4)}$

(3) Find the inverse Laplace transform of the given functions by using the convolution theorem:

$F(s) = \frac{1}{s^4(s^2+1)}$; $\int_0^t \frac{1}{6}(t-\tau)^3 \sin\tau d\tau$

$F(s) = \frac{1}{(s+1)^2(s^2+4)}$. $\int_0^t (t-\tau)e^{-(t-\tau)} \frac{1}{2}\sin 2\tau d\tau$

From Forza Horizon 4




# Thank You !!!