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VV256 RC2

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Contents

Picard Existence and Uniqueness Theorem

Consider the IVP $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. If f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle containing (x_0, y_0).

Then $\exists \delta > 0$, such that this IVP has a unique solution over $(x_0 - \delta, x_0 + \delta)$.

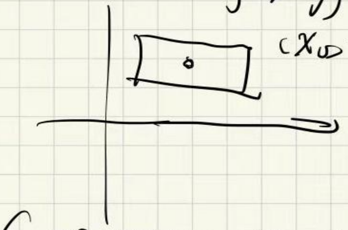
Remark: We don't even need the function $f(x, y)$ to be differentiable to y, we can simplify this condition to $|f(x, y_2) - f(x, y_1)| \leq L|y_2 - y_1|$, which is called **Lipschitz Condition**.

I am not sure whether this part will be tested because it is almost useless in your exam but maybe it can help cope with some extreme conditions and give you the confidence that the solution indeed exists.

Geometry Meaning (not required)

$\frac{dy}{dx} = f(x,y)$ $f(x,y) \to R$: $|x - x_0| \le a$, $|y - y_0| \le b$

$f(x,y)$

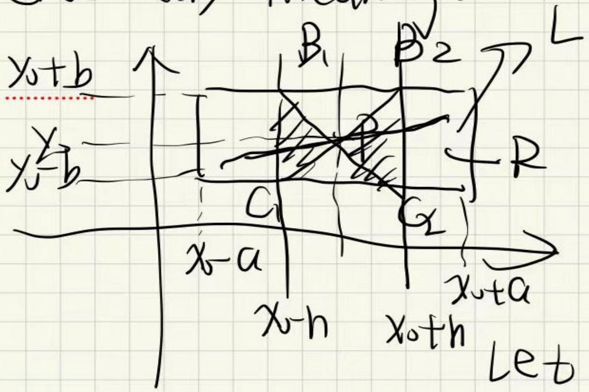


$\exists h > 0$

solution

$(x_0 - h, x_0 + h)$

Geometry meaning:



$P(x_0, y_0)$

$k_{B_1C_2} = -M$

$k_{B_2C_1} = M$

Let $h = \frac{b}{M}$

L: $y = \varphi(x)$

$|\varphi(x) - \varphi(x_0)| = |\varphi(x) - y_0| \le M|x - x_0| \le b$

来在域内

[常微分方程——一阶微分方程的解的存在唯一性定理 - 知乎 (zhihu.com)](zhihu.com)

Banach Contraction Mapping Principle

Complete Space The Space that any Cauchy Sequence defined in the space must be converged to the space.

Given a space $\mathbb{R}^n$, and a closed set D inside the space. If for any two points y_1, y_2 in D, we have a map $J : D \longrightarrow D$ which satisfies $|J(y_2) - J(y_1)| \leq \theta |y_2 - y_1|$ in which $\theta \in (0, 1)$, then the map J has a unique fixed point. That is to say there exist a unique point X which makes $J(X) = X$.

Peano Existence Theorem

Consider the IVP $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. If f is continuous in a rectangle containing (x_0, y_0), then IVP admits a solution in the rectangle (may be not unique, though).

eg: solution exist but not unique

initial condition in IVP

$\frac{dy}{dx} = 2\sqrt{y}$ (0, 0)

$$\Rightarrow y = \begin{cases} 0 & 0 \le x \le c \\ (x-c)^2 & c < x \le 1 \end{cases}$$

Remarks on these theorems

A continued first order (explicit) ODE satisfying Lipschitz Continuity $\longrightarrow$ existence and uniqueness of solution. (Picard Theorem)

A continued first order (explicit) ODE $\longrightarrow$ existence of solution (Peano Theorem).

However, A continued first order (explicit) ODE not satisfying Lipschitz Continuity does not necessarily indicate the existence and non-uniqueness of solution, as there exist weaker condition than Lipschitz Continuity. For example: Osgood Condition.

Picard Iteration

Picard Series $y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt (x \in I)$

Objective In one word, the method of Picard Iteration (Successive Approximation) is to construct a Picard Series for a given ODE and find something to which the series is convergent to.

Solution $y_n(x)$ when $n \longrightarrow \infty$

Q1

Solve IVP question: $y' = 2t(1+y)$ $y(0)=0$

by the method of successive approximation

$$\frac{dy}{dt} = 2t + 2ty$$

$dy = 2t(1+y)dt$ $\quad y - y_0 = \int_0^t 2x(1+y)dx$

$\Rightarrow$ picard series

$$y_{n+1} = y_0 + \int_0^t 2x(1+y_n(x))dx$$

$$y_1(t) = 0 + \int_0^t 2x\,dx$$

$$y_2(t) = \int_0^t 2x(1+x^2)dx = t^2 + \frac{1}{2}t^4$$

$$y_3(t) = \int_0^t 2x(1+x^2+\frac{1}{2}x^4)\,dx = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6$$

$\vdots$

$$y_n(t) = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots + \frac{1}{n!}t^{2n}$$

when $n \to \infty$

$$y_n(t) = e^{t^2} - 1$$

Second Order ODE

General Form $y'' + py' + qy = 0$

Feature Equation $y^2 + py + q = 0$

$\Delta > 0$: $y = C_1e^{y_1x} + C_2e^{y_2x}$

$\Delta = 0$: $y = (C_1 + C_2x)e^{y_1x}$

$\Delta < 0$: $y_1 = \alpha + \beta i$, $y_2 = \alpha - \beta i$, $y = e^{ax}(C_1cos\beta x + C_2sin\beta x)$

$y'' + 6y' + 5y = 0 \qquad y_1 = -1 \quad y_2 = -5$

$\Delta > 0 \qquad \Rightarrow y = C_1e^{-x} + C_2e^{-5x}$

Euler ODE (Second Order)

General Form $at^2y''(t) + bty'(t) + cy(t) = 0$

How to solve? Substitute $y(t) = t^\lambda$.

λ_1, λ_2 real and distinct: $y(t) = C_1 t_1^{\lambda_1} + C_2 t_2^{\lambda_2}$

$\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$: $y(t) = t^\alpha[C_1 cos(\beta lnt) + C_2 sin(\beta lnt)]$

$\lambda_1 = \lambda_2 = \frac{a-b}{2a}$: $y(t) = t_1^\lambda(C_1 + C_2 lnt)$

Higher Order? $a_n t^n \frac{d^n y}{dt^n} + a_{n-1} t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 t \frac{dy}{dt} + a_0 y = f(t)$

Q2

① $t^2y'' + 6ty' + 5y = 0$ $\quad y(t) = t^\lambda$

$\lambda(\lambda-1)t^\lambda + 6\lambda t^\lambda + 5t^\lambda = 0$ $\quad \lambda^2 + 5\lambda + 5 = 0$

$\lambda_{1,2} = \frac{-5 \pm \sqrt{5}}{2}$ $\quad y(t) = C_1 t^{\lambda_1} + C_2 t^{\lambda_2}$

② $t^2y'' + 6ty' + 9y = 0$ $\quad y(t) = t^\lambda$

$\lambda(\lambda-1)t^\lambda + 6\lambda t^\lambda + 9t^\lambda = 0$ $\quad \lambda^2 + 5\lambda + 9 = 0$

$\lambda_{1,2} = \frac{-5 \pm \sqrt{11}\, i}{2}$

$y(t) = t^{-\frac{5}{2}}\left[C_1 \cos\left(\frac{\sqrt{11}}{2}\ln t\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}\ln t\right)\right]$

Higher order

Homogeneous: Wronskian, Method of characteristic polynomial

Non-homogeneous: Method of variation of parameters, Method of coefficient comparison, Special form (Euler equation).

Wronskian

Consider two solutions $y_1(t)$ and $y_2(t)$ of the second-order linear homogeneous ODE:

$$y'' + a_1(t)y' + a_2(t)y = 0$$

$$W[y_1, y_2](t) = y_1 y_2' - y_2 y_1'$$

Then $\left(\frac{y_2}{y_1}\right)' = \frac{y_2' y_1 - y_1' y_2}{y_1^2} = \frac{W[y_1,y_2](t)}{y_1^2}$

$$y_2(t) = y_1(t)\left(C_1 \int \frac{exp(-\int a_1(\tau)d\tau)}{y_1^2(t)} + C_2\right).$$

Logic: $W = y_1 y_2' - y_1' y_2 = \left(\frac{y_2}{y_1}\right)' / y_1^2$

$W' = y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2 = y_1 y_2'' - y_1'' y_2$

原方程: $y'' + p(x) y' + q(x) y = 0 \quad y'' = -(py' + qy)$

$W' = y_1(-py_2' - qy_2) + y_2(py_1' + qy_1) = -py_1 y_2' + py_1' y_2$

$\frac{dW}{dx} = -pW \quad \dots \qquad = -pW$

Comment: Important! You need to memorize the Wronskian for 2nd order. However, for higher order, this get extremely complicated.

[阿贝尔微分方程恒等式 - 小时百科 (wuli.wiki)]()

Q3

(1) (2022 FA) Consider the differential equation $x^2y''+xy'+(x^2-\frac{1}{4})y=0, x>0$ with one solution given by $y_1=\frac{sinx}{\sqrt{x}}$. Find another independent solution.

$$x^2y''+xy'+(x^2-\tfrac{1}{4})y=0$$

$$y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=0 \qquad y_1=\frac{\sin x}{\sqrt{x}}$$

$$a_1(x)=\frac{1}{x}$$

$$W[y_1,y_2]=y_1y_2'-y_1'y_2$$

$$a_2(x)=1-\frac{1}{4x^2}$$

$$\frac{y_2(t)}{y_1(t)}=c_1\int\frac{\exp(-\int a_1(z)\,dz)}{y_1^2(t)}+C_2$$

$$=c_1\int\frac{\exp(-\int\frac{1}{x}dx)}{\frac{\sin^2x}{x}}+C_2$$

$$=c_1\int\frac{\exp(-\ln x)}{\frac{\sin^2x}{x}}+C_2=c_1\int\frac{1}{\sin^2x}+C_2$$

$$\frac{y_2}{y_1}=c_1\cot x+C_2 \quad \ldots$$

Method of Characteristic Polynomial

Assumption: If there is a solution $\alpha + \beta i$, there will be a solution $\alpha - \beta i$.

Solve: $a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$

Consider the feature equation: $a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 = 0$

Situation1: Distinct roots: $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \cdots + C_n e^{\lambda_n t}$

Complex: $\lambda = \alpha + \beta i \longrightarrow C e^{\alpha t} cos(\beta t)$, $\lambda = \alpha - \beta i \longrightarrow C e^{\alpha t} sin(\beta t)$

Situation2: Multiple roots:
$y(t) = C_1 e^{\lambda_1 t} + (C_2 + C_3 t + C_4 t^2) e^{\lambda_2 t} + \ldots$

Complex: $\lambda = \alpha + \beta i \longrightarrow (C_1 + C_2 t + C_3 t^2 + \ldots) e^{\alpha t} cos(\beta t)$,
$\lambda = \alpha - \beta i \longrightarrow (C_1 + C_2 t + C_3 t^2 + \ldots) e^{\alpha t} sin(\beta t)$

Q4

Let the characteristic equations have the following roots:

(1) $-3 \pm 2i, -1, 0, 2$

(2) $3, 3, 3, 0, 0, 1 \pm 3i$

(3) $2 \pm 5i, 2 \pm 5i, 1, 1, 1, 1, 3$

Write the solutions of the equations.

(1) $-3 \pm 2i$, -1, 0, 2

$$y(t) = C_1 e^{-t} + C_2 \boxed{e^{0t}} + C_3 e^{2t} + e^{-3t} \cdot [C_4 \sin 2t + C_5 \cos 2t]$$

(2) $3, 3, 3, 0, 0, 1 \pm 3i$

$$y(t) = e^{3t}[C_1 + C_2 x + C_3 x^2] + e^{0t}[C_4 + C_5 x] + e^{t}[C_6 \sin 3t + C_7 \cos 3t]$$

(3) $2 \pm 5i$ $2 \pm 5i$ $1, 1, 1, 1, 3$

$$y(t) = e^{2t}[\cos 5t\,(C_1 + C_2 t) + \sin 5t\,(C_3 + C_4 t)] + e^{t}[C_5 + C_6 t + C_7 t^2 + C_8 t^3] + C_9 e^{3t}$$

Method of variation of parameters

[“常数变易法”有效的原理-CSDN博客]

General Form: $y^{(n)}+a_1(t)y^{(n-1)}+\cdots+a_n(t)y=b(t)$

Step1: Find the solution to the homogeneous equation:
$y^{(n)}+a_1(t)y^{(n-1)}+\cdots+a_n(t)y=0$ ① 式1

Step2: Set up an equation system:

$$\begin{cases} y_1'C_1'(t)+y_2'C_2'(t)+\cdots+y_n'C_n'(t)=0 \\ y_1^{(2)}C_1'(t)+y_2^{(2)}C_2'(t)+\cdots+y_n^{(2)}C_n'(t)=0 \\ \vdots \\ y_1^{(n-1)}C_1'(t)+y_2^{(n-1)}C_2'(t)+\cdots+y_n^{(n-1)}C_n'(t)=b(t) \end{cases}$$

Step3: Solve for $C_i'(t)$ and integrate them.

The final solution is given by $y=C_1(t)y_1+C_2(t)y_2+\cdots+C_n(t)y_n$.

Q5

Use method of variation of parameters to solve the follow

$y''' - y' = \frac{e^x}{1+e^x}$

$y''' - y' = \frac{e^x}{1+e^x}$

① $y''' - y' = 0$ $\quad y^3 = y \quad y = 0, 1, -1$

$y_c(x) = C_1 + C_2 e^x + C_3 e^{-x}$

② $y''' - y' = F(x)$ Then C_1 C_2 C_3 from constant to function of x

$y = C_1(x) + C_2(x) e^x + C_3(x) e^{-x}$

若求完再代入 $y''' - y' = 0$ 发现满足题设

这里省略使用 step 2 的结论

$y_1(x) = 1 \quad y_2(x) = e^x \quad y_3 = e^{-x}$ $\qquad y_1$ 无用

$\cancel{y_1' C_1'} + y_2' C_2' + y_3' C_3' = 0$ $\qquad n = 2$

$\cancel{y_1'' C_1}$ $\quad y_2'' C_2' + y_3'' C_3' = \frac{e^x}{1+e^x}$

$$\begin{cases} e^x C_2' - e^{-x} C_3' = 0 \\ e^x C_2' + e^{-x} C_3' = \frac{e^x}{1+e^x} \end{cases}$$

$\ldots \quad C_2$

C_3

$\ldots$

Additional Exercises

$$x'' + x = \frac{1}{\cos t}$$

$$\Rightarrow \quad x^2 + 1 = 0 \qquad x = \pm i$$

$$x_c(t) = c_1 \cos t + c_2 \sin t$$

$$x(t) = c_1(t) \cos t + c_2(t) \sin t, \qquad K = 0$$

$$c_1'(-\sin t) + c_2' \cos t) = 0$$

$$c_1'(-\cos t) + c_2'(-\sin t) = \frac{1}{\cos t}$$

$$c_1'(t) = -\frac{\sin t}{\cos t} \qquad c_2'(t) = 1$$

$$c_1(t) = \ln|\cos t| + \gamma_1 \qquad c_2(t) = t + \gamma_2$$

$$x = \gamma_1 \cos t + \gamma_2 \sin t + \cos t \ln|\cos t| + t \sin t$$

Method of Coefficient comparison

When to use? Variation of parameters get not trivial.

General Form: $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = f(t)e^{\lambda t}$

Same steps when calculating the general solution y_c. For particular solution y_p, you need to look up the following table.

$f(t)e^{\lambda t}$	$y_p(t)$
$f(t)$	$t^k g(t)$
$f(t)e^{\alpha t}$	$t^k g(t)e^{\alpha t}$
$f(t)e^{\alpha t} sin\beta t$	$t^k e^{\alpha t}[g(t)sin\beta t + h(t)cos\beta t]$
$f(t)e^{\alpha t} cos\beta t$	$t^k e^{\alpha t}[g(t)sin\beta t + h(t)cos\beta t]$

Where k stands for the number of multiple roots, g(t) and h(t) stands for polynomials with the same order as f(t).

Last thing: $y = y_c + y_p$.

Q6

Use method of coefficient comparison to solve the following equation:

$$y'' + 3y' + 2y = 42e^{5x} + 390sin(3x) + 8x^2 - 2$$

$$y'' + 3y' + 2y = 42e^{5x} + 390\sin(3x) + 8x^2 - 2$$

$y^2 + 3y + 2 = 0 \qquad y_1 = -1 \quad y_2 = -2$

$y_c = C_1 e^{-x} + C_2 e^{-2x} \qquad y_p \sim \qquad k = 0$

$42e^{5x} \Leftrightarrow C_3 e^{5x}$

$390\sin(3x) \Leftrightarrow C_4 \sin 3x + C_5 \cos(3x)$

$8x^2 - 2 \Leftrightarrow C_6 x^2 + C_7 x + C_8$

求导后解方程 ⇒ 系数

It's your turn!!!

Additional case: (玩的)

$$y^{(6)} - 17y^{(5)} + 107y^{(4)} - 335y^{(3)} + 616y'' - 624y' + 240y = e^{t} + e^{4t} + e^{5t} + e^{6t}$$

五个重根为 1, 1, 4, 5, 1, 4　　1个2重根(4) 1个3重根

$$y_h = (c_1 + c_2x + c_3x^2)e^{x} + (c_4 + c_5x)e^{4x} + c_6e^{x}$$

$$y_p = t^3(c_7e^{t} + c_8) + t^2(c_9e^{4t} + c_{10}) + t(c_{11}e^{5t} + c_{12}) + c_{13}e^{6t}$$

↳待定系数

一般右侧 $e^{\lambda t}$ 形式

From Starfield

Thank You!!!