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上海交通大学密西根学院

VV256 RC1

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目 录

- Introduction
- Classification of ODE
- Simple cases
- Basic skills

Self Introduction

Contact me through email (shaox3@sjtu.edu.cn) , feishu or VX is OK. But please ask me questions before 10 pm on weekdays and 11 pm on weekends. It's time for me to play computer games.

Also, I'm not responsible for grading homework so I hold regular RC once a week and don't ask me questions about homework. If some of the exercises in the homework are interesting, I will select some.

For myself, I have been the TA of PHIL2650J, VP160 and STAT471. So there is no need to worry about my RC quality. And if I say something in Chinese sometimes, please don't report it to UEO or Pro. Cai. Indeed they cannot do anything to me.

Introduction of this course

This course will talk about basic methods for solving ODEs which is namely (Elementary Integration Methods), Existence and Uniqueness Theorem, Higher Order ODEs, ODEs system, eigenvalue problem, Laplace and Fourier Transform, Complex Analysis, Directional Derivative on Matrix functions.... This course is hard, hope you can survive it. (From TA Wang Boqian)

For myself, it is an applied course. All you need is just calculating and remembering some traditional examples. There is no need to master all contents. If you understand some parts completely and get full marks on one kind of question, the grades will be nice. For my studying in VV256, listening to the lecture is very useful and if you are lazy, searching basic information and skills online is also OK.

Tips

Sign up for the piazza.

Complete all the quiz sets on time, these can help you prepare for upcoming exams.

Copying other's assignments (from another group) is **strictly forbidden**, and Prof.Cai will put much emphasis on that this semester. The latex format…… I have no idea……

I still have no idea whether my RC should provide recordings because reading my RC slides is enough. It's concrete.

Warm-up beginning (From VP160 RC2)

Solve Easy Differential Equation

【题 3】 一质点以初速度 v_0 作直线运动，所受阻力与其速度的三次方成正比．试求质点速度和位置随时间的变化规律以及速度随位置的变化规律．

You may can solve such easy questions if you master the skills in VP150 and VP160

【解】 取质点运动所循的直线为 x 轴，取坐标原点 $x=0$ 为质点在 $t=0$ 时刻以初速度 v_0 开始运动的位置．由题设，质点的加速度为

$$a=\frac{\mathrm{d}v}{\mathrm{d}t}=-kv^3$$

或

$$\frac{\mathrm{d}v}{v^3}=-k\mathrm{d}t$$

How to classify the ordinary differential equation?

What	How
ODE & PDE	Number of independent variables
Order	Order of the differential equation
Linear & Nonlinear	Linearity
Homogeneous & Non-homogeneous	Homogeneousness
IVP & BVP	Condition

$$a = \frac{dv}{dt} = -kv^3$$

One variable: ODE
One order: First order
Linear :√
Homogeneous :√
IVP：？(Initial Value problems and Boundary Value problems)

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Example:

ODE & PDE

ODE: $\frac{dy}{dx} = xy$, $y'' + y' + 2y^2 = e^x$.

PDE: Heat equation: $u_x = \alpha u_{xx}$ (Will be covered later).

Order

First order: $y' = f(x, y)$ (Explicit form).

First order: $F(x, y, y') = 0$ (Implicit form).

Linear & Nonlinear

Nonlinear: $\frac{dy}{dx} \cdot y + 3 = x^2$, $y^2 + \frac{dy}{dx} = \sqrt{x}$.

Linear: $\frac{d^2y}{dx^2} + y = sin(\omega_0 x)(\omega_0 = \text{const})$,

$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$.

Homogeneous & Non-homogeneous

Homogeneous: $y' = f(\frac{y}{x})$.

Non-homogeneous: Cannot be represented in the above form.

IVP & BVP

Same equation, different conditions.

$y'' + y' + 2y^2 = e^x$.

IVP: $y(0) = 1, y'(0) = 2$. (We focus on IVP problems in VV256)

BVP: $y(0) = 1, y'(1) = 2$.

Simple case:

- Exact Equation
- Separate Equation
- First Order Linear ODE
- Homogeneous ODE
- Bernoulli Equation & Riccati Equation
- Method of Integrating Factors
- So on ……

Indeed, I almost never remember these forms. If your calculating ability is also strong, your can skip it and just use what we learned in VP150/VP160 to solve ODE questions. But if you are familiar with these equations, you can have better **mathematical intuition** to find the best way to solve it or make some perfect changes of variable.

Quick Review

- Chain Rule
- method of changing variables
- fractional substitution (分式代换)
- And so on
- method of elimination
- structured approach (informal saying)
- closed to “配凑”， “构造” in Chinese
- Guess !!! (similarity)
- method of undetermined coefficients

Exact Equation:

Expression: $P(x,y) + Q(x,y)y' = 0 \longrightarrow P(x,y)dx + Q(x,y)dy = 0$, where $P_y(x,y) = Q_x(x,y)$.

Step1: Check whether the equation is exact.

Step2: Find a function $F(x,y)$ such that its partial derivative with respect to x and y are P and Q.

Step3: $F(x,y) = C$ is the general solution.

Logic: suppose we have

$F(x,y) = C \Rightarrow \frac{d}{dx}[F(x, y(x))] = 0$

$F_x + F_y \frac{dy}{dx} = 0$

$F_x dx + F_y dy = 0 \Rightarrow$ $F_x = P(x,y)$, $F_y = Q(x,y)$

$\Rightarrow$ We get the orginal form

$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$

Like: $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$

Q1:

$$(ye^x + 2e^x + y^2)dx + (e^x + 2xy)dy = 0$$

Solution:

$P(x,y) = (ye^x + 2e^x + y^2), Q(x,y) = e^x + 2xy$

$P_y = Q_x = e^x + 2y$, so this equation is exact.

$F(x,y) = e^x y + xy^2 + 2e^x = C$, which is the general solution.

Easy ?

What if the former equation changes a little ?

Solve: $(e^x + \frac{2e^x}{y} + y)dx + (\frac{e^x}{y} + 2x)dy = 0$

$P(x,y) = e^x + \frac{2e^x}{y} + y$

$Q(x,y) = \frac{e^x}{y} + 2x$

$P_y = 2e^x(-\frac{1}{y^2}) + 1 \quad Q_x = \frac{1}{y}e^x + 2$

$P_y \neq Q_x$ not exact

How to recover this question to the former form?

In Chinese, “配凑”
Also we use Integrating Factor to do so

That’s why we need to have mathematical intuition

First rule of thumb:
Don’t leave algebraic fraction most time.

Q1-additional exercise

$$(3x^2y + 8xy^2)dx + (x^3 + 8x^2y + 12y^2)dy = 0, y(2) = 1$$

$P(x, y) = 3x^2y + 8xy^2$, $Q(x, y) = x^3 + 8x^2y + 12y$

$P_y = Q_x = 3x^2 + 16xy$, so this equation is exact.

$F(x, y) = x^3y + 4x^2y^2 + 4y^3 = C$, which is the general solution.

Plug in $y(2) = 1 \longrightarrow C = 28$.

Final solution: $F(x, y) = x^3y + 4x^2y^2 + 4y^3 = 28$.

Q1: useful table

$$ydx + xdy = d(xy)$$

$$\frac{ydx-xdy}{y^2} = d(\frac{x}{y})$$

$$\frac{ydx-xdy}{xy} = d(ln|\frac{x}{y}|)$$

$$\frac{ydx-xdy}{x^2+y^2} = d(arctan\frac{x}{y})$$

$$\frac{ydx-xdy}{x^2-y^2} = \frac{1}{2}d(ln|\frac{x-y}{x+y}|)$$

$$\frac{ydx+xdy}{x^2+y^2} = \frac{1}{2}d(ln(x^2+y^2))$$

Back to former questions:

$$(y^2dx + 2xydy) + (ye^x dx + e^x dy) + 2e^x dx = 0$$

$$y^2dx + 2xydy = d(xy^2)$$

$$ye^x dx + e^x dy = d(ye^x)$$

$$2e^x dx = d(2e^x)$$

To sum up, $d(xy^2 + ye^x + 2e^x) = 0 \longrightarrow xy^2 + ye^x + 2e^x = C$

Separate Equation:

Expression: $\frac{dy}{dx} = f(x)g(y)$

Step1: Discuss the condition of $g(y_0) = 0$ and find the equilibrium solution.

Step2: When $g(y_0) \neq 0, \frac{dy}{g(y)} = f(x)dx$

Step3: Integrate two sides.

Fundamental thought:
variables separation
“分离变量”

Q2: $\frac{dy}{dx} = x^m y^n$ where $m, n \neq 0$

Solution: check equilibrium solution

$y = 0$

$\begin{cases} \frac{1}{-n+1} y^{-n+1} \\ \frac{1}{m+1} x^{m+1} \end{cases}$

when $y \neq 0$ $\quad \frac{dy}{y^n} = x^m dx$

$\otimes \begin{cases} m = -1 \\ n = 1 \end{cases}$

① $m = -1 \quad n = 1 \quad \ln y = \ln x + C$

② $m = -1 \quad n \neq 1 \quad \frac{1}{-n+1} y^{-n+1} = \ln x + C$

③ $m \neq -1 \quad n = 1 \quad \ln y = \frac{1}{m+1} x^{m+1} + C$

④ $m \neq -1 \quad n \neq -1 \quad \frac{1}{-n+1} y^{-n+1} = \frac{1}{m+1} x^{m+1} + C$

Additional Exercises:

$$y\frac{dy}{dx} + (1 + y^2)sinx = 0, y(0) = 1$$

$\frac{y}{1+y^2}\frac{dy}{dx} = -sinx$

Integrate both sides: $\int \frac{ydy}{1+y^2} = -\int sinxdx$

General Solution: $y(x) = \pm(Ce^{-4sin^2\frac{x}{2}} - 1)^{\frac{1}{2}}, C \neq 0$

Plug in $y(0) = 1 \longrightarrow C = 2$

Final solution: $y(x) = (2e^{-4sin^2\frac{x}{2}} - 1)^{\frac{1}{2}}$

$\int \frac{x}{ax^2+b} dx$

$a=1$ $b=1$

$\int \frac{x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1) + C$

$\frac{1}{2} \ln(y^2+1) + C = \cos x$

$\ln(y^2+1) + C = 2\cos x$

$y^2+1 = e^{2\cos x - C}$

First Order ODE:

Expression: $y' + p(t)y = q(t)$

Solution (Without initial condition): $y(t) = \frac{1}{\mu(t)}(\int \mu(t)q(t)dt + C)$,

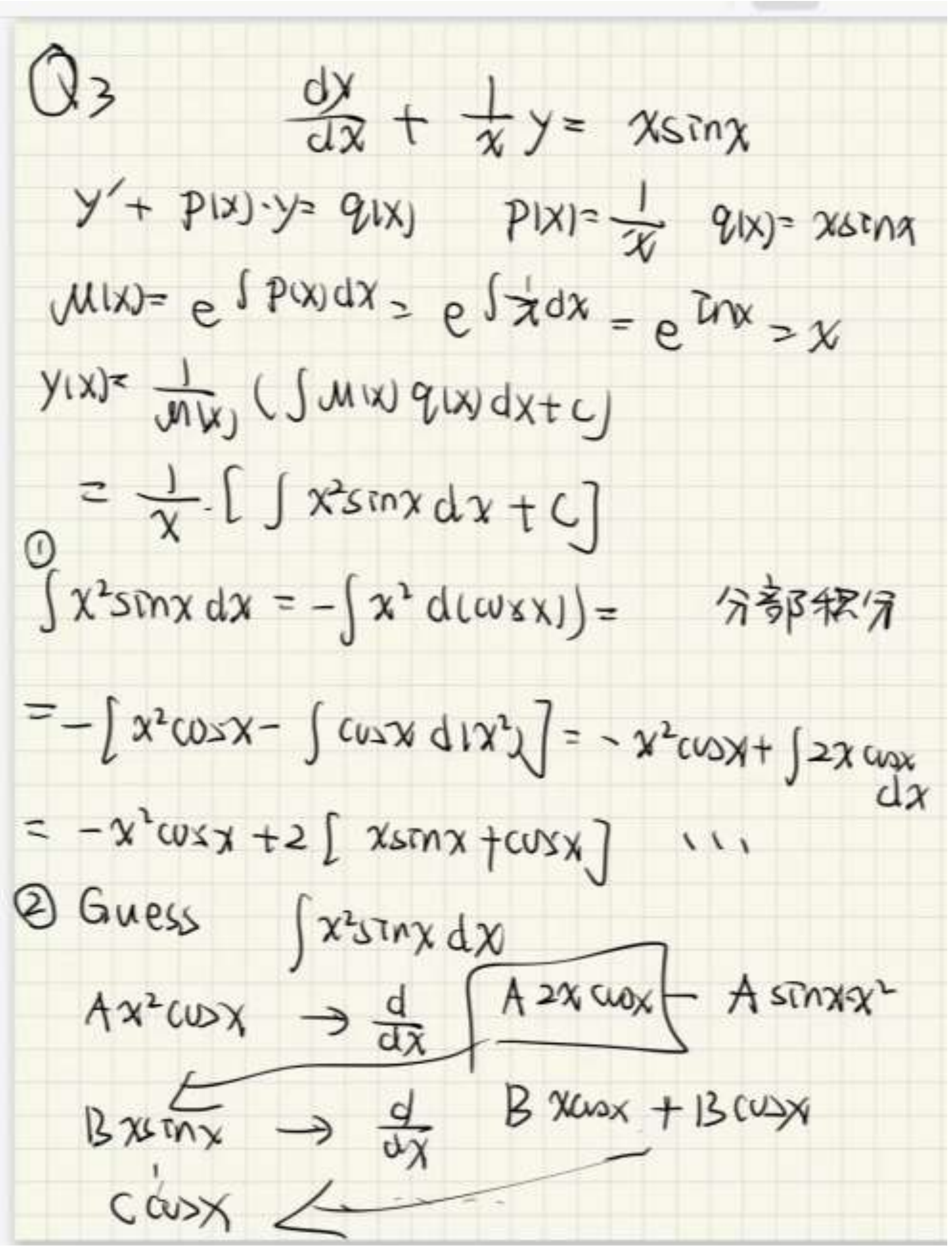
where $\mu(t) = e^{\int pdt}$.

Solution (With initial condition):

$y(t) = \frac{1}{\mu(t)}(\int_{t_0}^{t} \mu(\tau)q(\tau)d\tau + \mu(t_0)y(t_0))$.

[阿贝尔微分方程恒等式 - 小时百科 (wuli.wiki)]

If you want to see the proof, you can refer to this website or ask your VP160 classmates to show you the Lecture notes week 7 to see the whole proof though Prof. Qu's handwriting is dirty about this part. But it is clear and concrete.



Q3 $\frac{dy}{dx} + \frac{1}{x}y = x\sin x$

$y' + p(x)\cdot y = q(x)$ $\quad p(x) = \frac{1}{x}$ $\quad q(x) = x\sin x$

$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)dx + c\right)$

$= \frac{1}{x}\cdot\left[\int x^2\sin x\,dx + C\right]$

① $\int x^2\sin x\,dx = -\int x^2\,d(\cos x)) =$ 分部积分

$= -\left[x^2\cos x - \int \cos x\,d(x^2)\right] = -x^2\cos x + \int 2x\cos x\,dx$

$= -x^2\cos x + 2[x\sin x + \cos x]$ ...

② Guess $\int x^2\sin x\,dx$

$Ax^2\cos x \to \frac{d}{dx}$ $\quad A\,2x\cos x$ ← $A\sin x\,x^2$

$Bx\sin x \to \frac{d}{dx}$ $\quad Bx\cos x + B\cos x$

$C\cos x$

The calculating process of the form like :

$$\int x^n sinx \; or \int x^m \, cosx$$

is very important when we study Fourier Series but now we just have a quick look

Additional Exercises:

$$\frac{dy}{dx} = ysinx, y(0) = \frac{3}{2}$$

$p(x) = -sinx, q(x) = 0, \mu(x) = e^{\int p(x)dx} = e^{cosx}$

$y(x) = Ce^{-cosx}$

$y(0) = \frac{3}{2} \longrightarrow C = \frac{3}{2}e$

Final solution: $y(x) = \frac{3}{2}e^{1-cosx}$

Homogeneous ODE:

Expression: $\frac{dy}{dx} = f(x, y)$

Step1: Introduce a new variable $u = \frac{y}{x}$

Step2: Substitute $\frac{dy}{dx}$ with $\frac{du}{dx}$: $\frac{dy}{dx} = x\frac{du}{dx} + u$

Step3: Solve the Separable Equation: $x\frac{du}{dx} = f(1, u) - u.$

Q4:

$$x^2\frac{dy}{dx} = x^2 + xy + y^2$$

Transfer to the standard form of Homogeneous ODE:
$\frac{dy}{dx} = 1 + (\frac{y}{x}) + (\frac{y}{x})^2$
Let $u = \frac{y}{x} \longrightarrow x\frac{du}{dx} = 1 + u^2$
Transfer to the standard form of Separable ODE: $\frac{du}{1+u^2} = \frac{dx}{x}$.
Final solution: $arctan\frac{y}{x} = ln|x| + c$.

Additional Exercises:

$$\frac{dy}{dx} = \frac{x+y+1}{x-y+2}$$

Set the numerator and denominator to 0 respectively: $\alpha + \beta + 1 = 0$, $\alpha - \beta + 2 = 0 \longrightarrow \alpha = -\frac{3}{2}, \beta = \frac{1}{2}$

Let $\eta = y - \beta$, $\xi = x - \alpha \longrightarrow \frac{d\eta}{d\xi} = \frac{\xi+\eta}{\xi-\eta}$ (Similar to 4.2 now!)

Let $u = \frac{\eta}{\xi} \longrightarrow \frac{1-u}{1+u^2}du = \frac{d\xi}{\xi}$.

Integrate both sides: $arctanu - ln\sqrt{1+u^2} = ln|\xi| + C'$

Plug in x, y: $\frac{1}{2}ln[(x+\frac{3}{2})^2 + (y-\frac{1}{2})^2] = C + arctan(\frac{y-\frac{1}{2}}{x+\frac{3}{2}})$.

$x_0 = x + \frac{3}{2}$

$y_0 = y - \frac{1}{2}$

$u = \frac{y_0}{x_0}$

$\frac{dy_0}{dx_0} = \frac{1+\frac{y_0}{x_0}}{1-\frac{y_0}{x_0}}$

$\frac{dy_0}{dx_0} = x_0\frac{du}{dx_0} + u$

$x_0\frac{du}{dx_0} + u = \frac{1+u}{1-u}$

$x_0\frac{du}{dx_0} = \frac{1+u-u+u^2}{1-u} = \frac{1+u^2}{1-u}$

$\frac{1-u}{1+u^2}du = \frac{1}{x_0}dx$

Bernoulli Equation & Riccati Equation

Bernoulli Equation

Expression: $y' + p(t)y = q(t)y^n (n \neq 1)$

Step: Set $y = w^{\frac{1}{1-n}} \longrightarrow w' + (1-n)p(t)w = (1-n)q(t)$

Riccati Equation

Expression: $y' = q_0(t) + q_1(t)y + q_2(t)y^2$

Step1: Find a particular solution y_1

Step2: Set $y = y_1 + \frac{1}{w}$

Step3: The equation can be reduced to

$w' + (q_1 + 2q_2y_1)w = -q_2$

Hint: After calculating w, don't forget to calculate y!

$$\frac{dy}{dx} + p(t)y = q(t)y^n$$

$$\frac{1}{y^n}y' + \frac{1}{y^{n-1}}p(t) = q(t)$$

Let $w = \frac{1}{y^{n-1}} = y^{1-n} = y \cdot y^{-n}$

$$w' = (1-n)y^{-n}y' \qquad y^{-n} = \frac{w'}{1-n}\frac{1}{y'}$$

$$\frac{w'}{1-n} + p(t)w = q(t)$$

$$w' + (1-n)p(t)w = (1-n)q(t)$$

Seem strange but it is reasonable

Riccati: $y' + p(x)y^2 + q(x)y + r(x) = 0$

已知特定解 $y_0' + p(x)y_0^2 + q(x)y_0 + r(x) = 0$

$(y-y_0)' + p(x)(y^2-y_0^2) + q(x)(y-y_0) = 0$ $\quad y-y_0 = u$

$u' + (2p(x)y_0(x) + q(x))u + p(x)u^2 = 0$ ⇒ 伯努利

$u = w^{\frac{1}{1-2}}$ $\quad w = \frac{1}{u}$ ⇒ …… Logic

Q5: $y' + y = 2xy^2$ *(Mid1, 2022FA)*

Here $n = 2$, set $y = w^{-1}, p(x) = 1, q(x) = 2x$

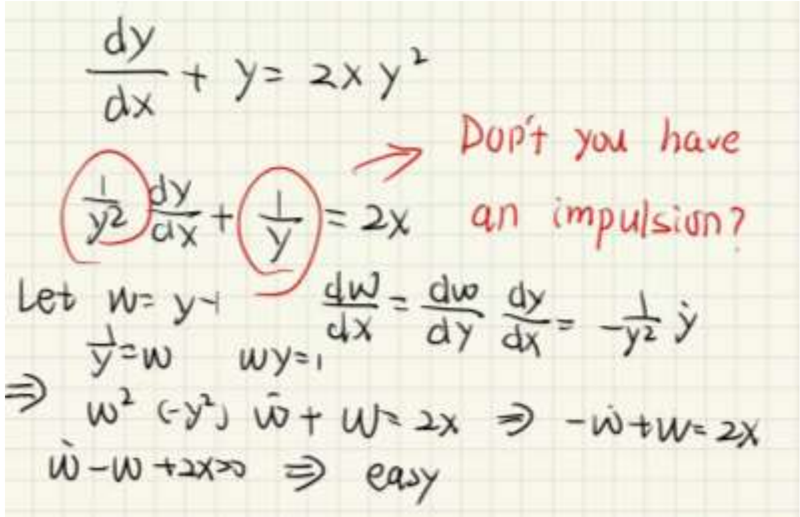
$w' - w = -2x$, which is a linear ODE.

According to case 3, $m(x) = -1, n(x) = -2x, \mu(x) = e^{-1dx} = e^{-x}$

$w(x) = e^x(\int -2x \cdot e^{-x}dx + C) = 2x + 2 + C \cdot e^x$

$y = \frac{1}{2x+2+C\cdot e^x}$ or $y = 0$ (Equilibrium solution).

Without the solution given before, this question should be solved by direct calculating and substituting. It is the method of changing variable



$\frac{dy}{dx} + y = 2xy^2$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = 2x$

Don't you have an impulsion?

Let $w = y^{-1}$ $\quad \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = -\frac{1}{y^2}\dot{y}$

$\frac{1}{y} = w \quad wy = 1$

$\Rightarrow w^2(-y^2)\dot{w} + w = 2x \Rightarrow -\dot{w} + w = 2x$

$\dot{w} - w + 2x = 0 \Rightarrow$ easy

Need Check

Q6:

$$y' = 1 + (y - x)^2, y(0) = \frac{1}{2}$$

$y' = (x^2 + 1) - 2xy + y^2$, which is a typical form of Riccati Equation.

Here $q_0(t) = x^2 + 1, q_1(t) = -2x, q_2(x) = 1$ repectively.

$w' + (-2x + 1)w = -1$, which is a linear ODE.

According to case 3,

$m(x) = -2x + 1, n(x) = -1, \mu(x) = e^{\int(-2x+1)dx} = e^{-x^2} + x$

$w(x) = e^{x^2} \cdot (\int -e^{-x^2} dx + C) = -\frac{\sqrt{\pi}}{2} erf(x) \cdot e^{x^2} + Ce^{x^2}$ ...

Plug in $y = y_1 + \frac{1}{w}$, $y = \frac{1}{2} + \frac{1}{-\frac{\sqrt{\pi}}{2} erf(x) \cdot e^{x^2} + Ce^{x^2}}$.

$$w(x) = \frac{1}{\mu(x)} \left(\int \mu(x) q(x) dx + c \right) = e^{x^2 - x} \left(\int e^{x - x^2} (-1) dx + c \right)$$

$$= e^{x^2 - x} \left[- \frac{e^{\frac{1}{4}} \sqrt{\pi} \, erf\left(\frac{2x-1}{2}\right)}{2} + c \right]$$

Rubbish question

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Direct Solution: $y' = 1 + (y-x)^2 \quad y(0) = \frac{1}{2}$

$y - x = t \quad y = x + t \quad \frac{dy}{dx} = 1 + \frac{dt}{dx}$

$1 + \frac{dt}{dx} = 1 + t^2 \quad t^2 = \frac{dt}{dx} \quad \frac{1}{t^2} dt = dx$

$-\frac{1}{t} = C + x \quad \frac{1}{x-y} = C + x$

$x - y = \frac{1}{C+x} \quad y = x - \frac{1}{C+x}$

$y(0) = \frac{1}{2} \quad y = 0 - \frac{1}{C} = \frac{1}{2} \quad C = -2$

$y = x - \frac{1}{x-2} = x + \frac{1}{2-x}$

considered as IVP can be solved

$y'(t) = t^2 + 2y(t) - y^2(t)$ 特解 $y(\frac{1}{2}) = \frac{1}{2}$

可能有问题，
自己凑的

$q_0(t) = t^2 \quad q_1(t) = 2 \quad q_2(t) = -1$

$w' + [2 + 2(-1)\frac{1}{2}]w = 1 \qquad w' + w = 1$

$w' = 1 - w \qquad \frac{dw}{dt} = 1 - w$

$\frac{dw}{1-w} = dt \qquad -\frac{d(1-w)}{1-w} = dt \qquad -\ln(1-w) = t + C$

$\ln(1-w) = -t + C'$

$1 - w = e^{-t+C'} \qquad w = 1 - Ce^{-t}$

$y(t) = \frac{1}{2} + \frac{1}{1 - Ce^{-t}}$ ...

Exact equation Continue —— Integrating Factor

Objective: Transforming non-exact equations to exact equations. $\left(\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \longrightarrow \frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}\right)$

Situation1: μ is only related to x, that is $\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \phi(x)$, whose integrating factor is $\mu = e^{\int \phi(x)dx}$.

Situation2: μ is only related to y, that is $\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P} = \phi(y)$, whose integrating factor is $\mu = e^{\int \phi(y)dy}$.

Q7

$$(3x^3 + y)dx + (2x^2y - x)dy = 0$$

First, check $P_y = 1, Q_x = 4xy - 1$, so this is not an exact equation.

$\frac{P_y - Q_x}{Q} = -\frac{2}{x}$, which is only related to x.

So $\mu = e^{\int \phi(x)dx} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$

Multiply two sides by $\frac{1}{x^2} \longrightarrow 3xdx + 2ydy + \frac{ydx - xdy}{x^2} = 0$

Grouping: $d(\frac{3}{2}x^2 + y^2 - \frac{y}{x}) = 0$

So the general solution is given by $\frac{3}{2}x^2 + y^2 - \frac{y}{x} = C$.

Additional Exercises

Sometimes the integrating factor can be identified directly.

$$\frac{dy}{dx} = -\frac{x}{y} + \sqrt{1+\left(\frac{x}{y}\right)^2}$$

$$x\,dx + y\,dy = \sqrt{x^2+y^2}\,dx$$

$$\frac{1}{2}\,d(x^2+y^2) = \sqrt{x^2+y^2}\,dx$$

easy: $\mu = \frac{1}{\sqrt{x^2+y^2}}$ $\times\mu$

$$\frac{d(x^2+y^2)}{2\sqrt{x^2+y^2}} \quad dx$$

$$\sqrt{x^2+y^2} = x + C \quad \ldots$$

Extension Part: implicit ODE of first order (may not be tested)

See another file

From A Plague Tale: Innocence

Thank you!