UM-SJTU JOINT INSTITUTE

VP160 RC5

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Contents

- Elasticity Model(not important)
- Gravitation and Celestial Motion
- Lagrange Mechanics

Elastic Model

Definition

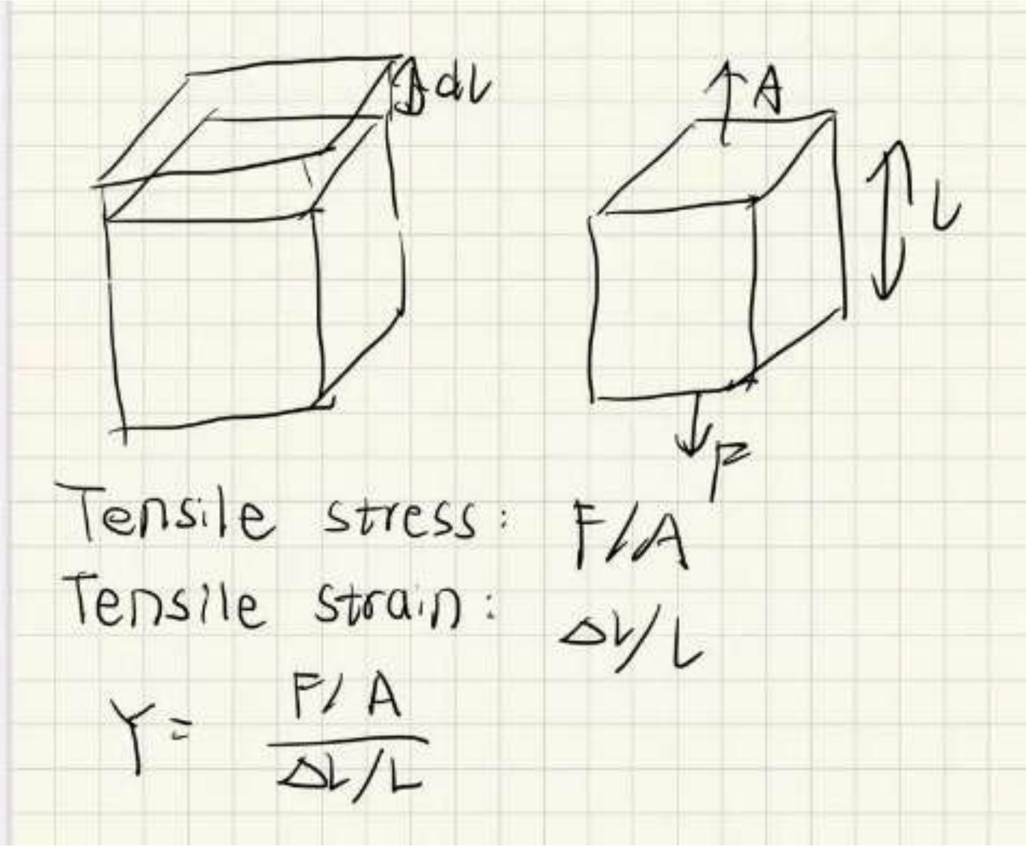
Real body deforms under external force.
Different from rigid body.

Stress and strain are proportional(Hooke's law)

Stress: force/unit area
Strain: fractional deformation due to stress:

$$Elastic\ modules = \frac{Stress}{strain}$$

Young’s modulus



This modulus implies stiffness of the material

The larger Young's modulus is, the more rigid the material is and the more difficult it is to deform. The smaller Young's modulus is, the softer the material is and the easier it is to deform. Young's modulus is a constant in the elastic deformation region of the stress-strain curve (usually the linear phase in which the stress and the strain become proportional).

Gravitation

Source: any mass

The gravitation is much weaker than the other three Fundamental interactions

$$F_{12} = -G\frac{m_1 m_2}{r_{12}2}\frac{r_{12}}{|r_{12}|}$$

$$G = 6.67 \times 10^{-11}[N \cdot \frac{m^2}{kg}]$$

Gravitational Field

$$E = \frac{F}{m} = -G\frac{M}{r^2}\frac{r}{|r|}$$

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Gravitational Potential Energy

We always set the infinite far place as zero potential.
Then the gravitational potential energy equals the work done by gravitation from infinite far place to now

$$U_p = -\frac{GMm}{r}$$

Then we can have first cosmic velocity, second cosmic velocity and the third cosmic velocity

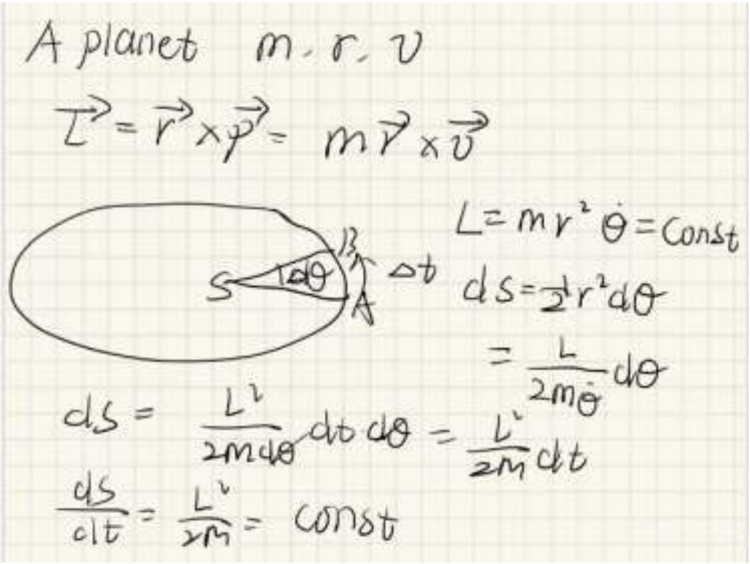
Kapler’s Law 1 and 2

1: Each planet moves in an elliptical orbit with the Sun at on of the focal point.

2:The line from the Sun to the given planet sweeps out equals areas in equal time.

If at time Δt, The area swept by the line between the planet and the sun in Δt is ΔA. Then this swept area is constant over any equal interval of time.

$\frac{\Delta A}{\Delta t}$ is constant



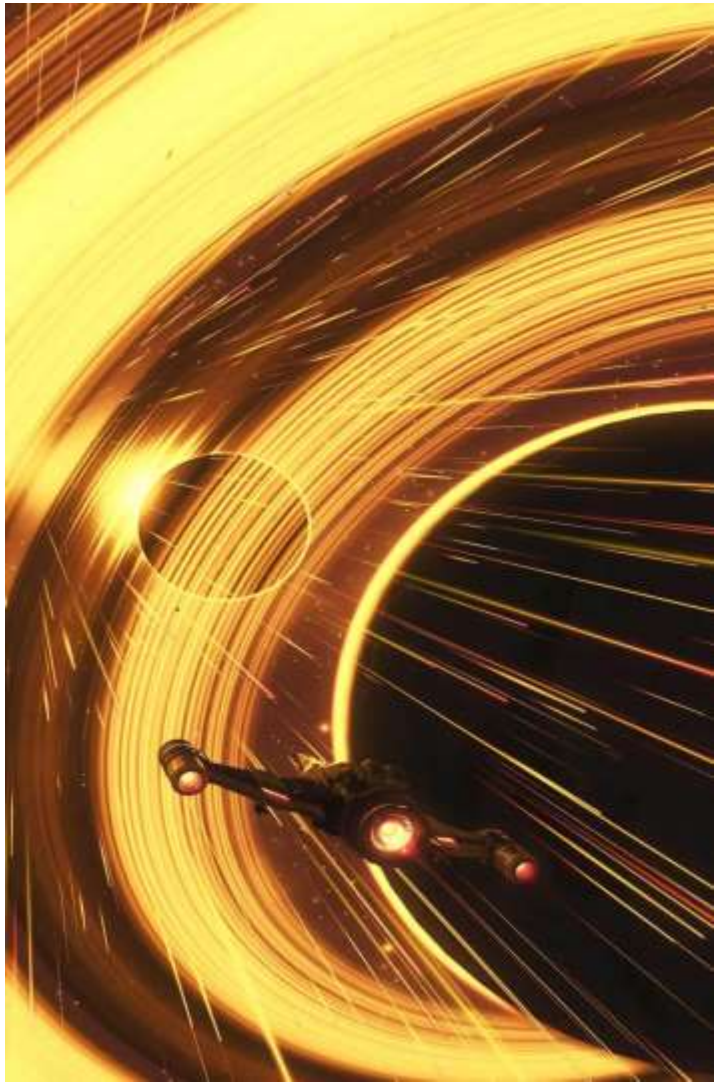
Kapler's Law 3

3:The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

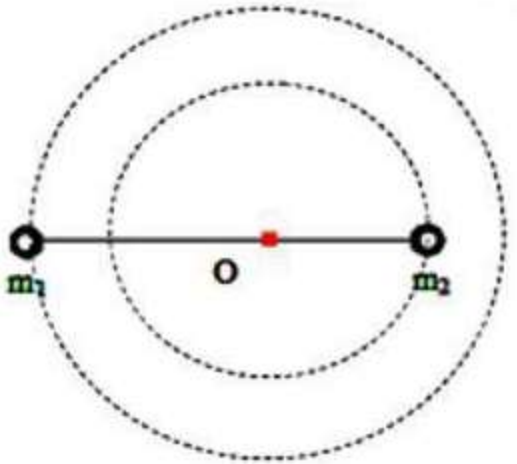
$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}}$$

$$\frac{R_1}{R_2} = \left(\frac{T_1}{T_2}\right)^{\frac{2}{3}}, \frac{T_1}{T_2} = \left(\frac{R_1}{R_2}\right)^{\frac{3}{2}}$$

Where, m1m2 is the mass of the two corresponding planets, T1T2 is the period of the two corresponding planets moving around the same star, and R1R2 is the average orbital radius of the two planets moving around the same star



Familiar Model：Two Star (双星问题)



Model Characteristics

- Two nearby celestial bodies revolve around a point on their connecting line (center of mass O) with the same angular velocity ω in uniform circular motion.

1. **Centripetal Force Balance**:
   - The centripetal force for circular motion of both celestial bodies is equal to the gravitational force between them.
   - Mathematical expression: $mR_m\omega^2 = MR_M\omega^2$
   - Therefore, $\frac{R_m}{R_M} = \frac{M}{m}$, where:
     - R_m and R_M are the distances from the bodies with masses m and M to the center of mass O, respectively.
2. **Equal Angular Velocity**:
   - Since both celestial bodies have the same angular velocity ω, their linear velocity ratio equals their distance ratio from the center of mass:
     $\frac{v_m}{v_M} = \frac{R_m}{R_M} = \frac{M}{m}$
3. **Centripetal Force Balance Derivation**:
   - The centripetal force for circular motion of both bodies is equal:
     $\frac{a_m}{a_M} = \frac{M}{m}$
     where a_m and a_M are the centripetal accelerations of the two bodies.
4. **Orbital Period Formula**:
   - Using the law of universal gravitation and the centripetal force formula, the orbital period T can be derived:
     $T = 2\pi\sqrt{\frac{L^3}{G(M+m)}}$
     where:
     - L is the distance between the two bodies,
     - G is the gravitational constant.

Common Conclusion: Density of planet

$$\frac{GMm}{R^2} = m\left(\frac{2\pi}{T}\right)^2 R$$

$$M = \frac{4\pi^2 R^3}{GT^2}$$

$$\rho = \frac{\frac{4\pi^2 R^3}{GT^2}}{\frac{4}{3}\pi R^3}$$

$$\rho = \frac{3\pi}{GT^2}$$

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Fundamental Knowledge about the curve

Ellipses, Hyperbolas, and Parabolas Unified Definition:

With a fixed point (focus) and a fixed line (directrix), the path of a point whose distance ratio to the focus and the directrix is a constant e.

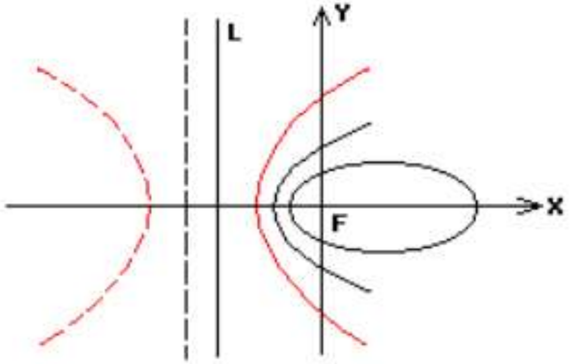
Using the left focus of an ellipse (or the right focus of a hyperbola, or the focus of a parabola) as the pole F, a polar coordinate system is established with the axis perpendicular to the directrix as the polar axis, and the perpendicular direction as K.

The unified polar coordinate equation for ellipses, hyperbolas, and parabolas is:

$$\rho = \frac{ep}{1 - e\cos\theta}$$

where ρ is the distance from the point to the directrix, $p > 0$.

- When $0 \leq e < 1$, the equation represents an ellipse.
- When $e > 1$, the equation represents a hyperbola; if $p > 0$, the equation represents only the right branch of the hyperbola, and if $p < 0$ is allowed, the equation represents the entire hyperbola.
- When $e = 1$, the equation represents a parabola opening to the right.



[圆锥曲线的极坐标方程 - 小时百科 (wuli.wiki)]

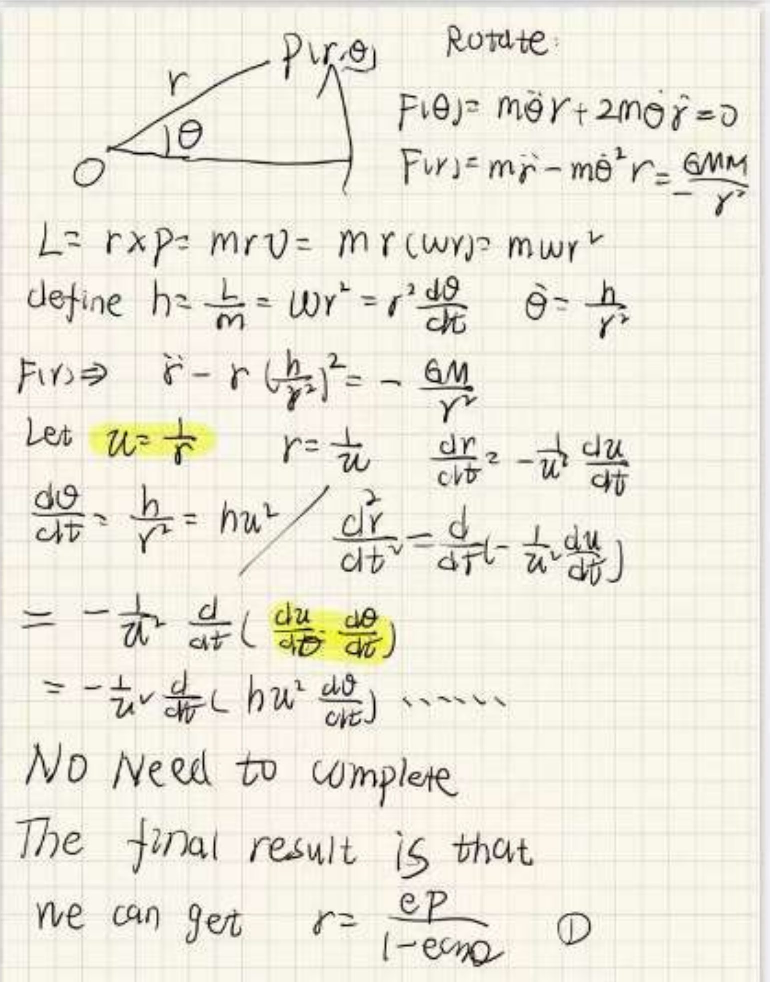
1+ecos and 1-ecos are both ok

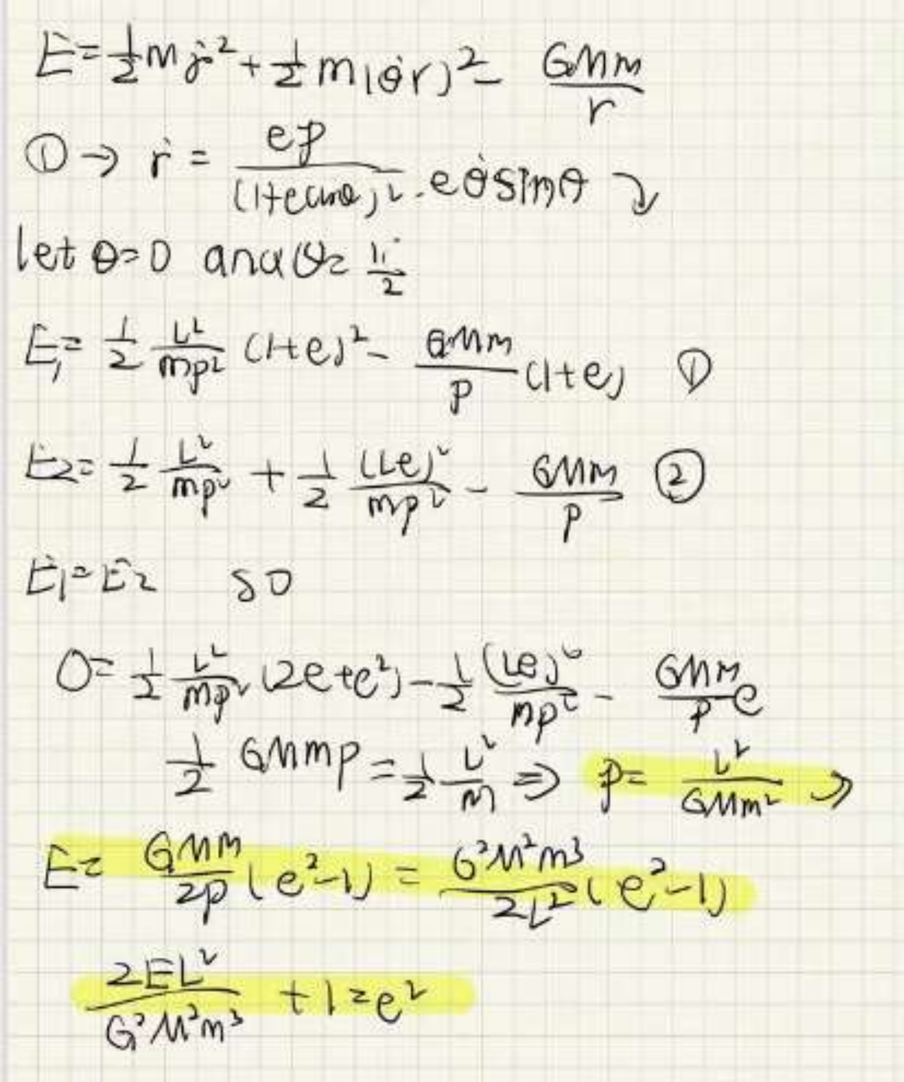
Important about the conclusion!!!

In the motion of the celestial body, the eccentricity of the orbit is related to the total energy, when $E < 0$, it is a closed elliptical orbit, when $E=0$, it is a parabola, and when $E > 0$, it is a hyperbola

The process is not acquired and the result can be remembered

Process (skip):





Exercises 1

Try to derive Newtonian gravitation from Kepler's first and second laws

Solution 1:

【题 13】

试由开普勒第一、二定律导出牛顿万有引力.

解 开普勒第二定律表明牛顿万有引力为有心力：即有

$$m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]=F(r),\ mr^2\frac{d\theta}{dt}=L(\text{守恒量}).$$

比纳公式：引入

$$u=\frac{1}{r},\ h=\frac{L}{m},$$

得

$$\frac{d\theta}{dt}=\frac{L}{mr^2}=hu^2,$$

$$\frac{dr}{dt}=\frac{dr}{d\theta}\frac{d\theta}{dt}=\frac{d}{d\theta}\left(\frac{1}{u}\right)hu^2=-u^{-2}\frac{du}{d\theta}\cdot hu^2=-h\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2}=\frac{d}{dt}\left(-h\frac{du}{d\theta}\right)=\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt}=-h\frac{d^2u}{d\theta^2}\cdot hu^2=-h^2u^2\frac{d^2u}{d\theta^2},$$

代入上面的径向动力学方程，得

$$\frac{F}{m}=-h^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}(hu^2)^2=-h^2u^2\frac{d^2u}{d\theta^2}-h^2u^3=-h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right),$$

即成比纳公式：

$$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{F}{m}.$$

由开普勒第一定律和比纳公式导出牛顿万有引力：

由开普勒第一定律和比纳公式导出牛顿万有引力：

椭圆轨道

Something Wrong here

$$r=\frac{p}{1+\varepsilon\cos\theta},\ u=\frac{1}{r}=\frac{(1+\varepsilon\cos\theta)}{p},$$

$$\frac{du}{d\theta}=-\frac{\varepsilon}{p}\sin\theta,$$

$$\frac{d^2u}{d\theta^2}=-\frac{\varepsilon}{p}\cos\theta,$$

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{p},\ h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{h^2u^2}{p}=\frac{h^2}{pr^2},$$

即得

$$F(r)=\frac{-mh^2}{pr^2}.$$

将施力者质量记为 M，引入参量 G，使得

$$F(r)=\frac{-GMm}{r^2},$$

则有

$$GM=\frac{h^2}{p}=\frac{L^2}{m^2p},$$

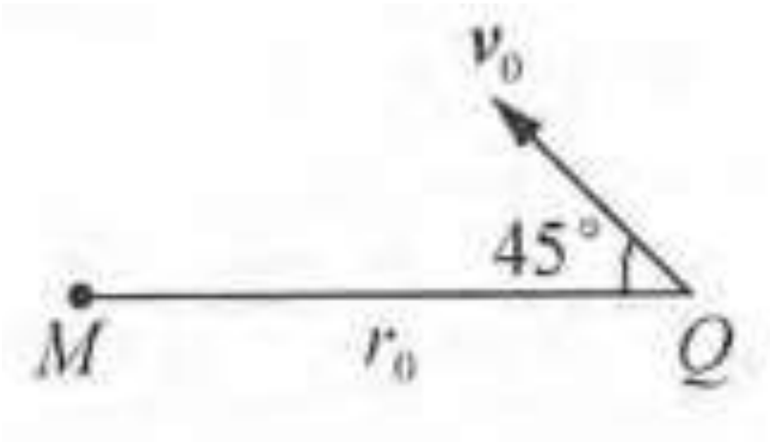
得

Exercises 2(skip)

The Sun is known to have a mass of M, and the asteroid Q is r away from the Sun with velocity v0

(1) When v0 is what value, Q's orbit is a parabola?

(2) If the orbit of Q is a parabola, try to find the maximum speed Vmax of Q in the process of motion

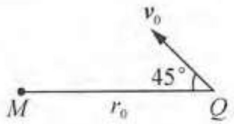


Solution 2:

已知太阳质量为 M，小行星 Q 与太阳相距 r_0 时速度 v_0 的方向如图所示.

（1）v_0 为多大值时，Q 的轨道是抛物线？

（2）若 Q 的轨道是抛物线，试求 Q 在运动过程中的最大速度 $v_{\max}$.



解 （1）由

$$\frac{1}{2}mv_0^2-G\frac{Mm}{r_0}=0$$

导得

$$v_0=\sqrt{\frac{2GM}{r_0}}.$$

（2）抛物线轨道若记为 $y^2=2px$，则太阳位于 $x=\frac{p}{2}$，$y=0$ 处，小行星位于抛物线顶点处的速度最大，此时 Q 与太阳相距$\frac{p}{2}$. 可列出下述能量方程和面积方程：

$$\frac{1}{2}mv_{\max}^2-G\frac{Mm}{\frac{p}{2}}=0,$$

$$\frac{1}{2}v_{\max}\cdot\frac{p}{2}=\frac{1}{2}v_0r_0\sin45^\circ=\frac{\sqrt{2}}{4}v_0r_0,$$

由此可得

$$v_{\max}^2p=4GM,\quad v_{\max}p=\sqrt{2}v_0r_0,$$

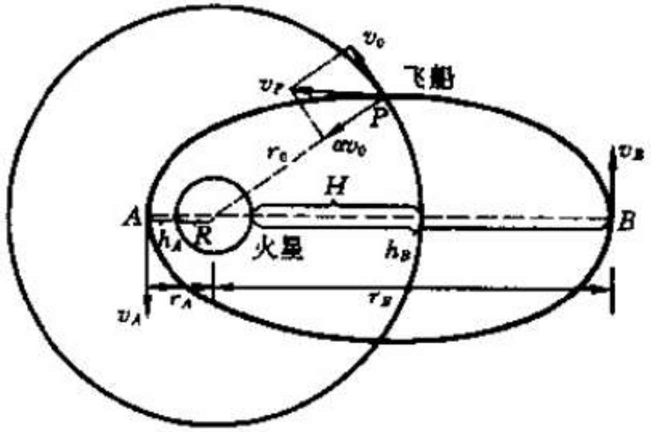
两式相除，即得

$$v_{\max}=\frac{4GM}{\sqrt{2}v_0r_0}=2\sqrt{\frac{GM}{r_0}}.\text{（相当于 } v_{\max}=\sqrt{2}v_0\text{）}$$

Exercises 3

As shown in the figure, a spaceship is orbiting Mars in a circular orbit with a speed of $v0$. It is known that the radius of Mars is R, and the height of the spaceship's circular orbit above the surface of Mars is H. Now, the spaceship fires a jet radially outward from the circular orbit in a very short period of time, so that the spaceship gains a radial velocity toward Mars of $av0$, where a is a constant much smaller than 1. Because the amount of gas ejected is very small, the mass of the spaceship can be considered unchanged after the jet. After the jet, the spaceship orbits Mars in a new elliptical orbit.

Find: 1. The height hA of the perigee of the spaceship's elliptical orbit from the surface of Mars and the height hB of the apogee of the elliptical orbit from the surface of Mars. 2. The orbital period of the spaceship in the elliptical orbit.



力图 4－11－1

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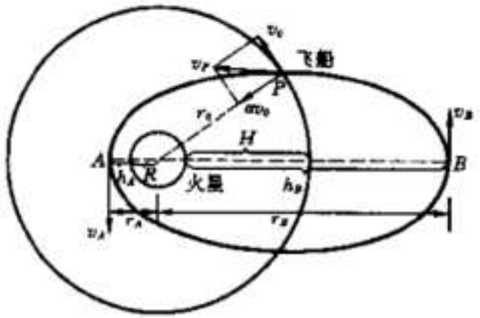
Solution 3:

【题 11】 如图，宇宙飞船绕火星沿圆轨道运行，运动速度为 v_0. 已知火星半径为 R，飞船圆轨道离火星表面的高度为 H. 今飞船在极短时间内，沿圆轨道径向向外侧点火喷气，使飞船获得指向火星的径向速度 αv_0，α 是远小于 1 的常数．因喷气量很小，喷气后飞船的质量可视为不变．喷气后，飞船绕火星沿新的椭圆轨道运行．

试求：1. 飞船椭圆轨道近火星点距火星表面的高度 h_A 以及远火星点距火星表面的高度 h_B．2. 飞船绕椭圆轨道的运行周期．

【分析】 飞船绕火星作圆运动时，圆轨道半径为 $r_0 = R + H$，面积速度为 $\frac{1}{2} r_0 v_0$. 飞船沿圆轨道径向向外侧喷气后，飞船将沿新的以火星为焦点的椭圆轨道运行，设近火星点为 A，远火星点为 B，A 和 B 与火星中心的距离分别为 r_A 和 r_B（$r_A = h_A + R$，$r_B = h_B + R$）. 飞船沿椭圆轨道运行时，面积速度恒定，机械能守恒．



力图 4－11－1

应特别注意，飞船沿圆轨道径向向外侧喷气后，使之获得指向火星的径向速度 αv_0，因喷气方向与飞船原先绕火星作圆轨道运动的速度 v_0 垂直，故喷气并不改变 v_0 的大小和方向，喷气后飞船的速度 v_p 是 v_0 与 αv_0（其方向与 v_0 垂直）的矢量和．换言之，$v_0 = v_p \sin\theta$，故短时间喷气后，飞船绕椭圆轨道的面积速度为 $\frac{1}{2} r_0 v_p \sin\theta = \frac{1}{2} r_0 v_0$，就等于飞船喷气前绕圆轨道运行的面积速度，亦即喷气前后飞船的轨道虽然改变了，但面积速度不变．由飞船沿椭圆轨道运行时，面积速度恒定及机械能守恒两条规律可以解出 r_A 和 r_B，再由开普勒第三定律可求出飞船绕椭圆轨道的运行周期．

Solution 3:

【解】 设火星和飞船的质量分别为 M 和 m，飞船沿椭圆轨道运行时，飞船与火星中心的距离统一用 r 表示，飞船的速度统一用 v 表示，再加下标注明飞船所在位置．

根据分析，飞船喷气前绕圆轨道运行的面积速度 $\frac{1}{2}r_0v_0$ 等于喷气后飞船绕椭圆轨道运行在 P 点的面积速度 $\frac{1}{2}r_0v_p\sin\theta$（$P$ 点是圆和椭圆的交点）．由开普勒第二定律，后者又应等于飞船在近火星点 A 和远火星点 B 的面积速度 $\frac{1}{2}r_Av_A$ 和 $\frac{1}{2}r_Bv_B$．故有

$$\frac{1}{2}r_0v_0=\frac{1}{2}r_0v_p\sin\theta=\frac{1}{2}r_Av_A=\frac{1}{2}r_Bv_B$$

即

$$r_0v_0=r_Av_A=r_Bv_B \tag{1}$$

由机械能守恒定律，有

$$\frac{1}{2}mv_p^2-\frac{GMm}{r_p}=\frac{1}{2}mv_A^2-\frac{GMm}{r_A}=\frac{1}{2}mv_B^2-\frac{GMm}{r_B}$$

因

$$v_p^2=v_0^2+(\alpha v_0)^2,\quad r_p=r_0$$

由以上三式，有

$$\frac{1}{2}m[v_0^2+(\alpha v_0)^2]-\frac{GMm}{r_0}=\frac{1}{2}mv_A^2-\frac{GMm}{r_A}=\frac{1}{2}mv_B^2-\frac{GMm}{r_B} \tag{2}$$

飞船沿原圆轨道运行时，有

$$\frac{GMm}{r_0^2}=\frac{mv_0^2}{r_0}$$

即

$$GM=r_0v_0^2 \tag{3}$$

联立(1)、(2)、(3)式，可得出关于 r（指 r_A 或 r_B）的二次方程式为

$$(1-\alpha^2)r^2-2r_0r+r_0^2=0$$

上式有两个解，大者就是 r_B，小者就是 r_A，为

$$\begin{cases} r_A=\dfrac{r_0}{1+\alpha}=\dfrac{R+H}{1+\alpha} \\ r_B=\dfrac{r_0}{1-\alpha}=\dfrac{R+H}{1-\alpha} \end{cases}$$

r_A 和 r_B 分别是飞船近火星点和远火星点与火星中心的距离．故近火星点和远火星点距火星表面的高度分别为

$$\begin{cases} h_A=r_A-R=\dfrac{H-\alpha R}{1+\alpha} \\ h_B=r_B-R=\dfrac{H+\alpha R}{1-\alpha} \end{cases}$$

设飞船椭圆轨道的半长轴为 a，则

$$r_A+r_B=2a$$

即

$$a=\frac{1}{2}(r_A+r_B)=\frac{r_0}{1-\alpha}$$

飞船喷气前绕圆轨道运行的周期为

$$T_0=\frac{2\pi r_0}{v_0}$$

设飞船喷气后绕椭圆轨道运行的周期为 T，由开普勒第三定律，有

$$\frac{T}{T_0}=\left(\frac{a}{r_0}\right)^{3/2}$$

故

$$T=T_0\left(\frac{a}{r_0}\right)^{3/2}=\frac{2\pi r_0}{v_0}\left(\frac{1}{1-\alpha^2}\right)^{3/2}$$

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Lagrange Mechanics

Lagrangian

The most significant quantity in Lagrangian Mechanics

$$L = K - U$$

Euler-Lagrange Equation

For $i = 1, 2, \ldots, f$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

I think this part will not be difficult in the final exam.

Fundamental Process

The steps for applying Lagrange's equation for a conservative force system are as follows: (1) Analyze the constraints of the system, such as whether the system is a complete system, and determine the degrees of freedom of the mechanical system. (2) Select the same number of generalized coordinates as the degrees of freedom. (3) Use generalized coordinates and generalized velocities to express the kinetic energy T of the mechanical system, and use generalized coordinates to express the potential energy V. Then write the Lagrangian function of the system L = T - V. (4) List the Lagrange equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial q_a}\right) - \frac{\partial L}{\partial q_a} = 0$ (where a = 1, 2, ..., s), the group of second-order ordinary differential equations of s q_a is the kinetic equation of the complete system. (5) Solve the equation and discuss it.

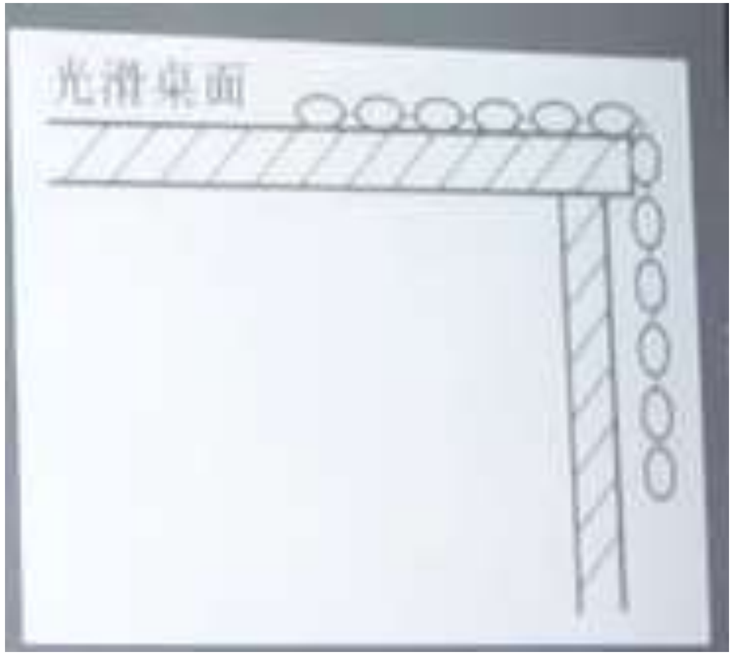
3 应用步骤

对于保守力系的问题，应用拉格朗日方程的步骤如下：(1) 分析系统所受的约束，如系统确为完整系统，确定力学体系的自由度；(2) 选取与自由度相同数目的广义坐标；(3) 用广义坐标和广义速度表示出力学体系的动能 T，用广义坐标表示出势能 V，并进而写出体系的拉格朗日函数 $L=T-V$；(4) 列出拉格朗日方程：$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0$（其中 $\alpha=1,2,\cdots,s$），s 个 q_α 的二阶常微分方程组就是完整系统的动力学方程；(5) 解方程并讨论。

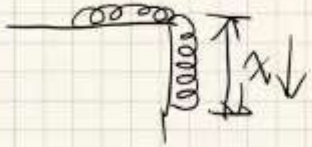
[谈拉格朗日方程在高中物理竞赛中的应用_孙伟 - 道客巴巴 (doc88.com)]

Simple question:

A homogeneous chain, length l, half on a smooth table, the other half hanging outside the table. The chain has no initial speed. What is the speed of the chain when it all leaves the table?



Solution:



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$L = L(x, \dot{x}, t)$

$L = K - U$

$K = \frac{1}{2}m\dot{x}^2$

$V = -\frac{m}{L}x\, g\, \frac{x}{2} = -\frac{mg}{2L}x^2$

$L = K - V = \frac{1}{2}m\dot{x}^2 + \frac{mg}{2L}x^2$

$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m \cdot 2 \cdot \dot{x} = m\dot{x}$

$\frac{\partial L}{\partial x} = \frac{mg}{2L} \cdot 2x = \frac{mg}{L}x$

$\frac{d}{dt}(m\dot{x}) = \frac{mg}{L}x$

$m\ddot{x} = \frac{mg}{L}x \qquad \ddot{x} = \frac{g}{L}x$ ① remember solution

How to solve $\dot{x}$ directly?

② $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x} \cdot \frac{d\dot{x}}{dx}$

$\dot{x}\frac{d\dot{x}}{dx} = \frac{g}{L}x \Rightarrow \int \dot{x}\, d\dot{x} = \int \frac{g}{L}x\, dx$

$\frac{1}{2}(\dot{x})^2 = \left[\frac{1}{2}(x)^2 + C\right]\frac{g}{L}$

$x = \frac{L}{2} \quad \dot{x} = 0 \Rightarrow$

we get $C = -\frac{1}{8}L^2$

$\Rightarrow (\dot{x})^2 = ((x)^2 + 2C)\frac{g}{L}$

$= \frac{g}{L}x^2 - \frac{gL}{4}$

$x = L \quad v = \dot{x} = \sqrt{\frac{3}{4}gL}$

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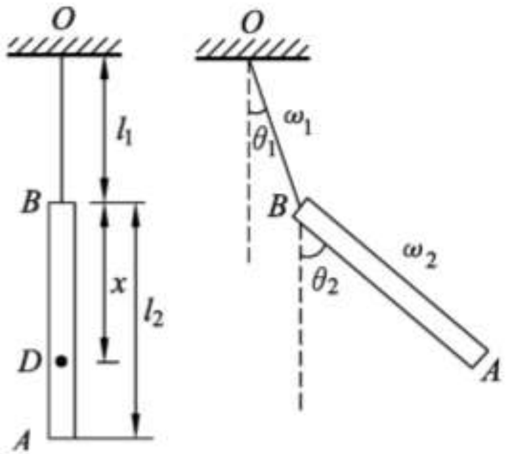
Interesting Question:

As shown in Figure a, AB is a uniform thin rod with a mass of m and a length of l. The upper end B of the rod is suspended from a fixed point O by an inextensible soft rope with a length of l. Initially, both the rope and the rod are hanging vertically. All subsequent movements occur in the same vertical plane.

(1) Now, a momentary impulse I is applied horizontally to the point D on the rod. If, in the instant after the impulse is applied, the angular velocity of point B rotating around the suspension point is the same as the angular velocity of the rod rotating around its center of mass, find the distance x from point D to point B and the initial angular velocity ω of point B rotating around the suspension point.



(2) Assume that at a certain moment, the angles between the rope and the rod and the vertical direction are θ_1 and θ_2 respectively (as shown in Figure b), and the angular velocities of the rope rotating around the fixed point O and the rod rotating around its center of mass are ω_1 and ω_2 respectively. Find the angular accelerations α_1 and α_2 of the rope rotating around the fixed point O and the rod rotating around its center of mass.

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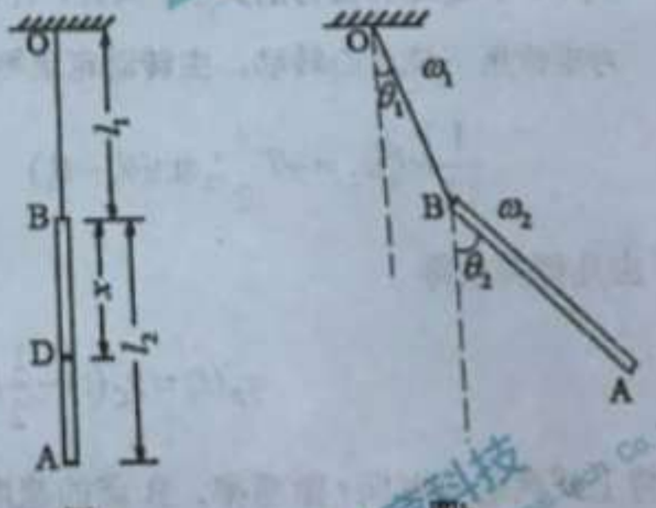
Chinese Tradition:

2.（15 分）如图 a，AB 为一根匀质细杆，质量为 m，长度为 l_2；杆上端 B 通过一不可伸长的软轻绳悬挂到固定点 O，绳长为 l_1。开始时绳和杆均静止下垂。此后所有运动均在同一竖直面内。

（1）现对杆上的 D 点沿水平方向施加一瞬时冲量 I，若在施加冲量后的瞬间，B 点绕悬点 O 转动的角速度和杆绕其质心转动的角速度相同，求 D 点到 B 点的距离 x 和 B 点绕悬点 O 转动的初始角速度 ω_0。

（2）设在某时刻，绳和杆与竖直方向的夹角分别为 θ_1 和 θ_2（如图 b 所示），绳绕固定点 O 和杆绕其质心转动的角速度分别为 ω_1 和 ω_2，求绳绕固定点 O 和杆绕其质心转动的角加速度 α_1 和 α_2。



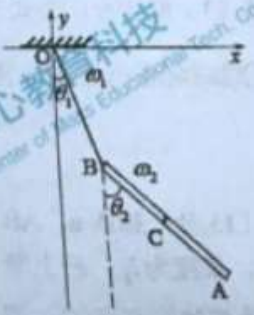
图a　图b

From thirty-third physics final competition

[第三十三届全国中学生物理竞赛决赛理论试题及答案 (eduzhixin.com)](eduzhixin.com)

Normal Solution 1:

（2）解法一：水平竖直方向坐标系如图，在 O、B、A 三点所在竖直平面内，以 O 为原点、水平向右的射线为 x 轴、竖直向上的射线为 y 轴，建立平面坐标系。



杆的质心 C 的加速度(a_{Cx},a_{Cy})满足质心运动定理

$$ma_{Cx}=-T\sin\theta_1,\quad ma_{Cy}=-mg+T\cos\theta_1 \qquad (7)$$

式中，T 是绳的张力的大小。同时，杆在绳张力 T 相对于杆的质心的力矩作用下绕质心转动，由转动定理得

$$\frac{1}{12}ml_2^2\alpha_2=-T\frac{1}{2}l_2\sin(\theta_2-\theta_1) \qquad (8)$$

由几何关系得

$$x_B(t)=x_C(t)-\frac{1}{2}l_2\sin\theta_2(t),\quad y_B(t)=y_C(t)+\frac{1}{2}l_2\cos\theta_2(t) \qquad (9)$$

将上式两边对时间 t 微商得，B 点的速度满足条件

$$v_{Bx}(t)=v_{Cx}(t)-\frac{1}{2}\omega_2(t)l_2\cos\theta_2(t),\quad v_{By}(t)=v_{Cy}(t)-\frac{1}{2}\omega_2(t)l_2\sin\theta_2(t) \qquad (10)$$

将上式两边对时间 t 微商得，B 点的加速度满足条件

$$a_{Bx}=a_{Cx}-\frac{1}{2}\alpha_2l_2\cos\theta_2+\frac{1}{2}\omega_2^2l_2\sin\theta_2,\quad a_{By}=a_{Cy}-\frac{1}{2}\alpha_2l_2\sin\theta_2-\frac{1}{2}\omega_2^2l_2\cos\theta_2 \qquad (11)$$

同时 B 点随不可伸长的绳绕 O 点做定轴转动，应有 (12)

$$x_B(t)=l_1\sin\theta_1(t),\quad y_B(t)=-l_1\cos\theta_1(t)$$

将上式两边对时间 t 微商得，B 点的速度还满足条件 (13)

$$v_{Bx}(t)=\omega_1(t)l_1\cos\theta_1(t),\quad v_{By}(t)=\omega_1(t)l_1\sin\theta_1(t)$$

将上式两边对时间 t 微商得，B 点的加速度还满足条件 (14)

$$a_{Bx}=\alpha_1l_1\cos\theta_1-\omega_1^2l_1\sin\theta_1,\quad a_{By}=\alpha_1l_1\sin\theta_1+\omega_1^2l_1\cos\theta_1$$

【或避开 B 点，直接得 C 点的位置坐标

$$x_C(t)=l_1\sin\theta_1(t)+\frac{1}{2}l_2\sin\theta_2(t),\quad y_C(t)=-l_1\cos\theta_1(t)-\frac{1}{2}l_2\cos\theta_2(t) \qquad (9)(12)$$

$$\dot{x}_C(t)=l_1\dot{\theta}_1(t)\cos\theta_1(t)+\frac{1}{2}l_2\dot{\theta}_2(t)\cos\theta_2(t),\quad \dot{y}_C(t)=l_1\dot{\theta}_1(t)\sin\theta_1(t)+\frac{1}{2}l_2\dot{\theta}_2(t)\sin\theta_2(t) \qquad (10)(13)$$

$$a_{Cx}(t)=-l_1\left(\alpha_1\cos\theta_1-\omega_1^2\sin\theta_1\right)-\frac{1}{2}l_2\left(\alpha_2\cos\theta_2-\omega_2^2\sin\theta_2\right),$$

$$a_{Cy}(t)=l_1\left(\sin\theta_1\,\alpha_1+\cos\theta_1\,\omega_1^2\right)+\frac{l_2}{2}\left(\sin\theta_2\,\alpha_2+\cos\theta_2\,\omega_2^2\right) \qquad (11)(14)$$

】

联立(7)(8)(11)(14)式，可解得绳绕悬点和杆绕质心的角加速度分别为

$$\alpha_1=-\frac{2g[\sin\theta_1+3\cos\theta_2\sin(\theta_1-\theta_2)]+3l_1\omega_1^2\sin[2(\theta_1-\theta_2)]+4l_2\omega_2^2\sin(\theta_1-\theta_2)}{2l_1\left[1+3\sin^2(\theta_1-\theta_2)\right]},$$

$$\alpha_2=\frac{12g\cos\theta_1\sin(\theta_1-\theta_2)+12l_1\omega_1^2\sin(\theta_1-\theta_2)+3l_2\omega_2^2\sin[2(\theta_1-\theta_2)]}{2l_2\left[1+3\sin^2(\theta_1-\theta_2)\right]} \qquad (15)$$

Normal Solution 2:

解法二：沿杆坐标系 。

在 O、B、A 三点所在竖直平面内，以质心 C 为原点、沿杆斜向上的射线为 y 轴、竖直于杆斜向右的射线为 x 轴，建立平面坐标系。

杆的质心 C 的加速度 (a_{Cx}, a_{Cy}) 满足质心运动定理

$$ma_{Cx} = -mg\sin\theta_2 + T\sin(\theta_2-\theta_1),\quad ma_{Cy} = -mg\cos\theta_2 + T\cos(\theta_2-\theta_1) \qquad ⑦$$

式中，T 是绳的张力的大小。同时，杆在绳张力 T 相对于杆的质心的力矩作用下绕质心转动，

由转动定理得

$$\frac{1}{12}ml_2^2\alpha_2 = -T\frac{1}{2}l_2\sin(\theta_2-\theta_1) \qquad ⑧$$

由几何关系得 B 点的加速度为

$$a_{Bx} = l_1\alpha_1\cos(\theta_2-\theta_1) + l_1\omega_1^2\sin(\theta_2-\theta_1),\quad a_{By} = -l_1\alpha_1\sin(\theta_2-\theta_1) + l_1\omega_1^2\cos(\theta_2-\theta_1) \qquad ⑭$$

由几何关系还可得 B 点的加速度满足条件

$$a_{Bx} = a_{Cx} - \frac{1}{2}l_2\alpha_2,\quad a_{By} = a_{Cy} - \frac{1}{2}l_2\omega_2^2 \qquad ⑪$$

【或由几何关系得 C 点的加速度

$$a_{Cx} = a_{Bx} + \frac{1}{2}l_2\alpha_2,\quad a_{Cy} = a_{By} + \frac{1}{2}l_2\omega_2^2 \qquad ⑪$$

】

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Normal Solution 3:

解法三：沿绳坐标系。

在 O、B、A 三点所在竖直平面内，以 O 为原点、沿绳斜向上的射线为 y 轴、竖直于绳斜向右的射线为 x 轴，建立平面坐标系。

杆的质心 C 的加速度(a_{Cx},a_{Cy})满足质心运动定理

$$ma_{Cx}=-mg\sin\theta_1,\quad ma_{Cy}=-mg\cos\theta_1+T \qquad ⑦$$

式中，T 是绳的张力的大小。同时，杆在绳张力 T 相对于杆的质心的力矩作用下绕质心转动。

由转动定理得

$$\frac{1}{12}ml_2^2\alpha_2=-T\frac{1}{2}l_2\sin(\theta_2-\theta_1) \qquad ⑧$$

由几何关系得 B 点的加速度为

$$a_{Bx}=l_1\alpha_1,\quad a_{By}=l_1\omega_1^2 \qquad ⑭$$

由几何关系还可得 B 点的加速度满足条件

$$a_{Cx}=a_{Bx}+\frac{1}{2}l_2\alpha_2\cos(\theta_2-\theta_1)-\frac{1}{2}l_2\omega_2^2\sin(\theta_2-\theta_1),$$

$$a_{Cy}=a_{By}+\frac{1}{2}l_2\alpha_2\sin(\theta_2-\theta_1)+\frac{1}{2}l_2\omega_2^2\cos(\theta_2-\theta_1) \qquad ⑪$$

【或由几何关系得 C 点的加速度

$$a_{Cx}=a_{Bx}+\frac{1}{2}l_2\alpha_2\cos(\theta_2-\theta_1)-\frac{1}{2}l_2\omega_2^2\sin(\theta_2-\theta_1),$$

$$a_{Cy}=a_{By}+\frac{1}{2}l_2\alpha_2\sin(\theta_2-\theta_1)+\frac{1}{2}l_2\omega_2^2\cos(\theta_2-\theta_1) \qquad ⑪$$

】

【对于此坐标系，还可写做：

$$\frac{1}{12}ml_2^2\alpha_2=-T\frac{1}{2}l_2\sin(\theta_2-\theta_1) \qquad ⑧$$

$$-mg\cos\theta_1+T=l_1\omega_1^2+\frac{1}{2}l_2\alpha_2\sin(\theta_2-\theta_1)+\frac{1}{2}l_2\omega_2^2\cos(\theta_2-\theta_1) \qquad (a1)$$

$$-mg\sin\theta_1=l_1\alpha_1+\frac{1}{2}l_2\alpha_2\cos(\theta_2-\theta_1)-\frac{1}{2}l_2\omega_2^2\sin(\theta_2-\theta_1) \qquad (a2)$$

由⑧(a1)式可消去 T，解得 α_2，带入(a2)可解得 α_1。这是最为简便的方法。

】

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Normal Solution 4:

解法四：非惯性参考系。

在 O、B、A 三点所在竖直平面内，以 O 为原点、沿绳斜向上的射线为 y 轴、竖直于绳斜向右的射线为 x 轴，建立平面坐标系。取以 O 点为参考点的惯性参考系，记为 O 系；取以 B 点为参考点的非惯性参考系，记为 B 系。

在惯性系 O 系中杆的质心 C 的加速度 (a_{Cx}, a_{Cy}) 满足质心运动定理

$$ma_{Cx} = -mg\sin\theta_1, \quad ma_{Cy} = -mg\cos\theta_1 + T \qquad ⑦$$

式中，T 是绳的张力的大小。同时，杆在绳张力 T 相对于杆的质心的力矩作用下绕质心转动，由转动定理得

$$\frac{1}{12}ml_2^2\alpha_2 = -T\frac{1}{2}l_2\sin(\theta_2-\theta_1) \qquad ⑧$$

由几何关系知，非惯性系 B 系相对于惯性系 O 系的加速度为

$$a_{Bx} = l_1\alpha_1, \quad a_{By} = l_1\omega_1^2 \qquad ⑭$$

杆质心 C 在非惯性系 B 系中的加速度为

$$a_{Cx}' = \frac{1}{2}l_2\alpha_2\cos(\theta_2-\theta_1) - \frac{1}{2}l_2\omega_2^2\sin(\theta_2-\theta_1),$$

$$a_{Cy}' = \frac{1}{2}l_2\alpha_2\sin(\theta_2-\theta_1) + \frac{1}{2}l_2\omega_2^2\cos(\theta_2-\theta_1) \qquad ⑪\text{-}1$$

从而惯性系 O 系中质心 C 的加速度可表示为

$$a_{Cx} = a_{Bx} + a_{Cx}', \quad a_{Cy} = a_{By} + a_{Cy}' \qquad ⑪\text{-}2$$

【或者考虑非惯性力，在非惯性系 B 系中运用牛顿第二定律得到：

$$-mg\cos\theta_1 + T - l_1\omega_1^2 = \frac{1}{2}l_2\alpha_2\sin(\theta_2-\theta_1) + \frac{1}{2}l_2\omega_2^2\cos(\theta_2-\theta_1) \qquad (a1)$$

$$-mg\sin\theta_1 - l_1\alpha_1 = \frac{1}{2}l_2\alpha_2\cos(\theta_2-\theta_1) - \frac{1}{2}l_2\omega_2^2\sin(\theta_2-\theta_1) \qquad (a2)$$

与解法三类似，由⑧(a1)式可消去 T，解得 α_2，带入(a2)可解得 α_1。

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All Untidy , Right?

Tidy Solution（use Lagrange rule）

解析：(1) 设在施加冲量后的瞬间杆的质心 C 速度为 v_C，由冲量定理得

$$I=mv_C. \tag{1}$$

由刚体转动定理得

$$I\left(x-\frac{l_2}{2}\right)=\frac{ml_2^{\,2}}{12}\omega_0. \tag{2}$$

B、C 点以同一角速度绕 O 点转动，B 点速度满足

$$v_B=\omega_0 l_1=v_C-\omega_0\frac{l_2}{2}. \tag{3}$$

由(1)(2)(3)式得

$$x=\left[\frac{1}{2}+\frac{l_2}{6(2l_1+l_2)}\right]l_2,\qquad \omega_0=\frac{2I}{m(2l_1+l_2)}. \tag{4}$$

(2) 系统有 2 个自由度，以绳和杆与竖直方向的夹角 θ_1 和 θ_2 作为系统的广义坐标. 建立以点 O 为原点，水平向右为 x 轴，竖直向下为 y 轴的坐标系，则杆的质心坐标为

$$x_C=l_1\sin\theta_1+\frac{l_2}{2}\sin\theta_2,\qquad y_C=l_1\cos\theta_1+\frac{l_2}{2}\cos\theta_2. \tag{5}$$

(5)式对时间求导得质心速度

$$v_{Cx}=l_1\omega_1\cos\theta_1+\frac{l_2}{2}\omega_2\cos\theta_2,\qquad v_{Cy}=-l_1\omega_1\sin\theta_1-\frac{l_2}{2}\omega_2\sin\theta_2, \tag{6}$$

主动力为杆的重力，是保守力. 系统的动能为

$$T=\frac{1}{2}\left[\left(l_1\omega_1\cos\theta_1+\frac{l_2}{2}\omega_2\cos\theta_2\right)^2+\left(-l_1\omega_1\sin\theta_1-\frac{l_2}{2}\omega_2\sin\theta_2\right)^2\right]+\frac{1}{2}\frac{ml_2^{\,2}}{12}\omega_2^2. \tag{7}$$

系统的势能

$$V=-mg\left(l_1\cos\theta_1+\frac{l_2}{2}\cos\theta_2\right). \tag{8}$$

拉格朗日函数

$$L=T-V. \tag{9}$$

由拉格朗日方程得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\frac{\partial L}{\partial\theta_1}=\frac{1}{2}ml_1[2l_1\alpha_1+l_2\alpha_2\cos(\theta_1-\theta_2)+l_2\omega_2^2\sin(\theta_1-\theta_2)+2g\sin\theta_1]=0. \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\frac{\partial L}{\partial\theta_2}=\frac{1}{6}ml_2[2l_2\alpha_2+3l_1\alpha_1\cos(\theta_1-\theta_2)-3l_1\omega_1^2\sin(\theta_1-\theta_2)+3g\sin\theta_2]=0. \tag{11}$$

由(10)、(11)式得

$$\alpha_1=-\frac{2g[\sin\theta_1+3\cos\theta_2\sin(\theta_1-\theta_2)]+3l_1\omega_1^{\,2}\sin[2(\theta_1-\theta_2)]+4l_2\omega_2^{\,2}\sin(\theta_1-\theta_2)}{2l_1[1+3\sin^2(\theta_1-\theta_2)]}. \tag{12}$$

$$\alpha_2=\frac{12g\cos\theta_1\sin(\theta_1-\theta_2)+12l_1\omega_1^{\,2}\sin(\theta_1-\theta_2)+3l_2\omega_2^{\,2}\sin[2(\theta_1-\theta_2)]}{2l_2[1+3\sin^2(\theta_1-\theta_2)]}. \tag{13}$$

From Red Dead Redemption 2