UM-SJTU JOINT INSTITUTE

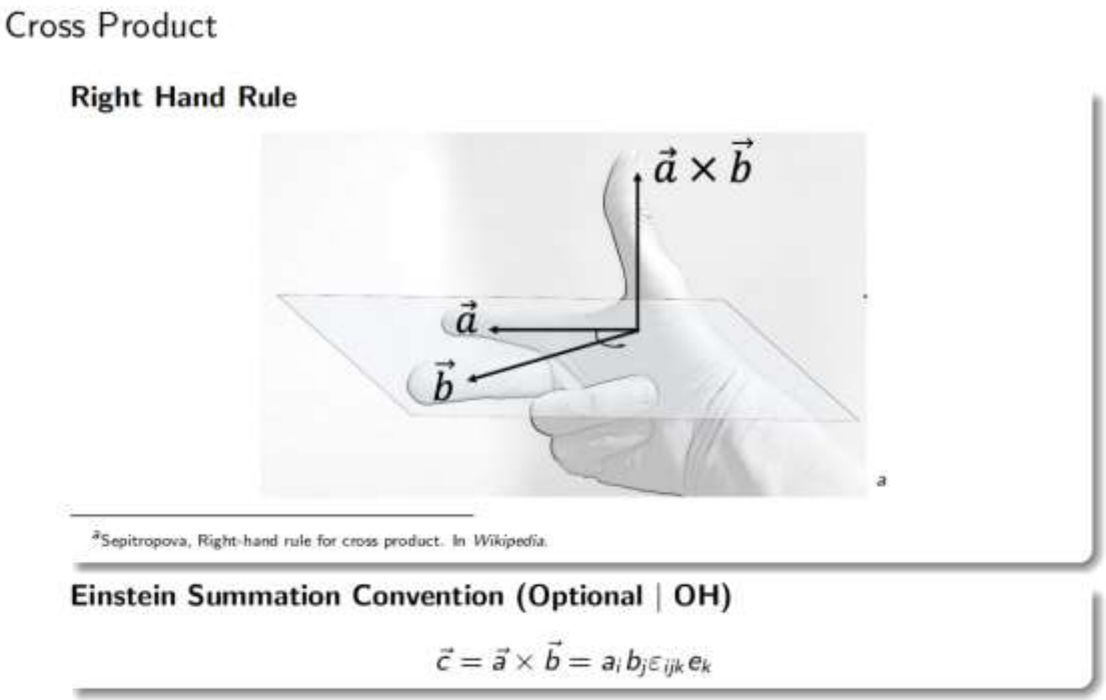
VP160 Mid Big RC Part1

Shao
Yujie

Contents

- Fundamental math definition about kinematics and coordinate system
- Newton's laws and dynamics
- Simple energy and momentum

Basic Operational Rule



Cross Product

Right Hand Rule

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[a] Sepitropova, Right-hand rule for cross product. In *Wikipedia*.

Einstein Summation Convention (Optional | OH)

$$\vec{c} = \vec{a} \times \vec{b} = a_i b_j \varepsilon_{ijk} e_k$$

$$t = r \times F$$

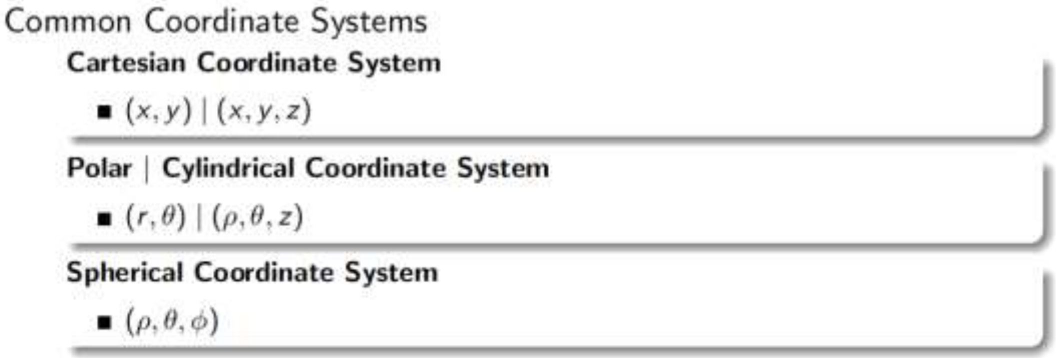
$$L = r \times p$$

Tensor: A tensor relates two(in general non-parallel) vectors, its mathematical structure is that of matrix

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$$

$$a \times (b \times c) = (a \cdot c) \cdot b - (a \cdot b) \cdot c$$

Basic Coordinate System



Common Coordinate Systems

Cartesian Coordinate System

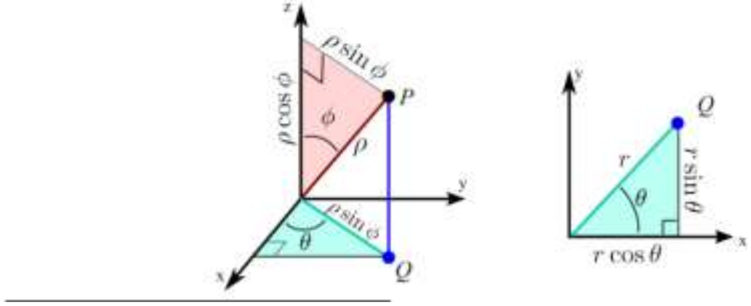
- $(x, y) \mid (x, y, z)$

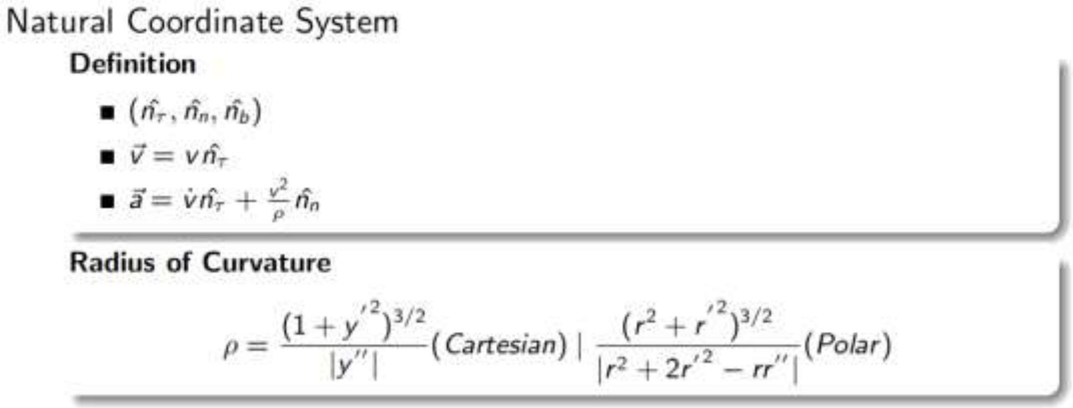
Polar | Cylindrical Coordinate System

- $(r, \theta) \mid (\rho, \theta, z)$

Spherical Coordinate System

- (ρ, θ, ϕ)





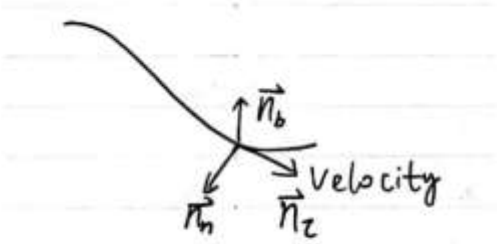
Natural Coordinate System

Definition

- $(\hat{n}_\tau, \hat{n}_n, \hat{n}_b)$
- $\vec{v} = v\hat{n}_\tau$
- $\vec{a} = \dot{v}\hat{n}_\tau + \frac{v^2}{\rho}\hat{n}_n$

Radius of Curvature

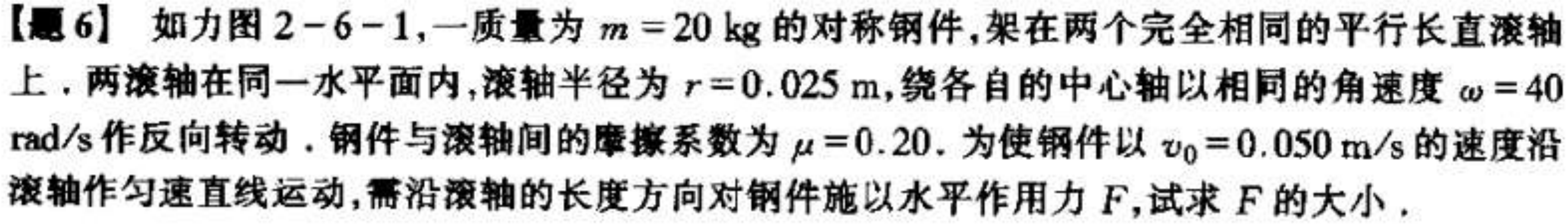
$$\rho = \frac{(1+y'^2)^{3/2}}{|y''|}(Cartesian) \mid \frac{(r^2+r'^2)^{3/2}}{|r^2+2r'^2-rr''|}(Polar)$$



相对运动 $v = v' + v0$

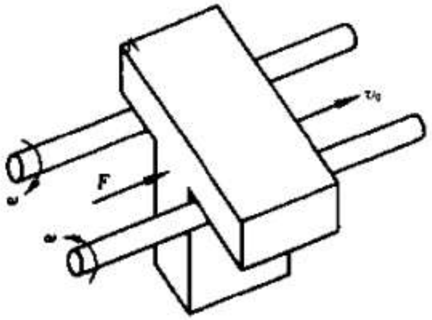
Relative Motion(almost without any physics knowledge)

You just need your eyes



【题6】 如力图 2－6－1，一质量为 $m=20\ \mathrm{kg}$ 的对称钢件，架在两个完全相同的平行长直滚轴上．两滚轴在同一水平面内，滚轴半径为 $r=0.025\ \mathrm{m}$，绕各自的中心轴以相同的角速度 $\omega=40\ \mathrm{rad/s}$ 作反向转动．钢件与滚轴间的摩擦系数为 $\mu=0.20$．为使钢件以 $v_0=0.050\ \mathrm{m/s}$ 的速度沿滚轴作匀速直线运动，需沿滚轴的长度方向对钢件施以水平作用力 F，试求 F 的大小．

It is in the midterm exam last year



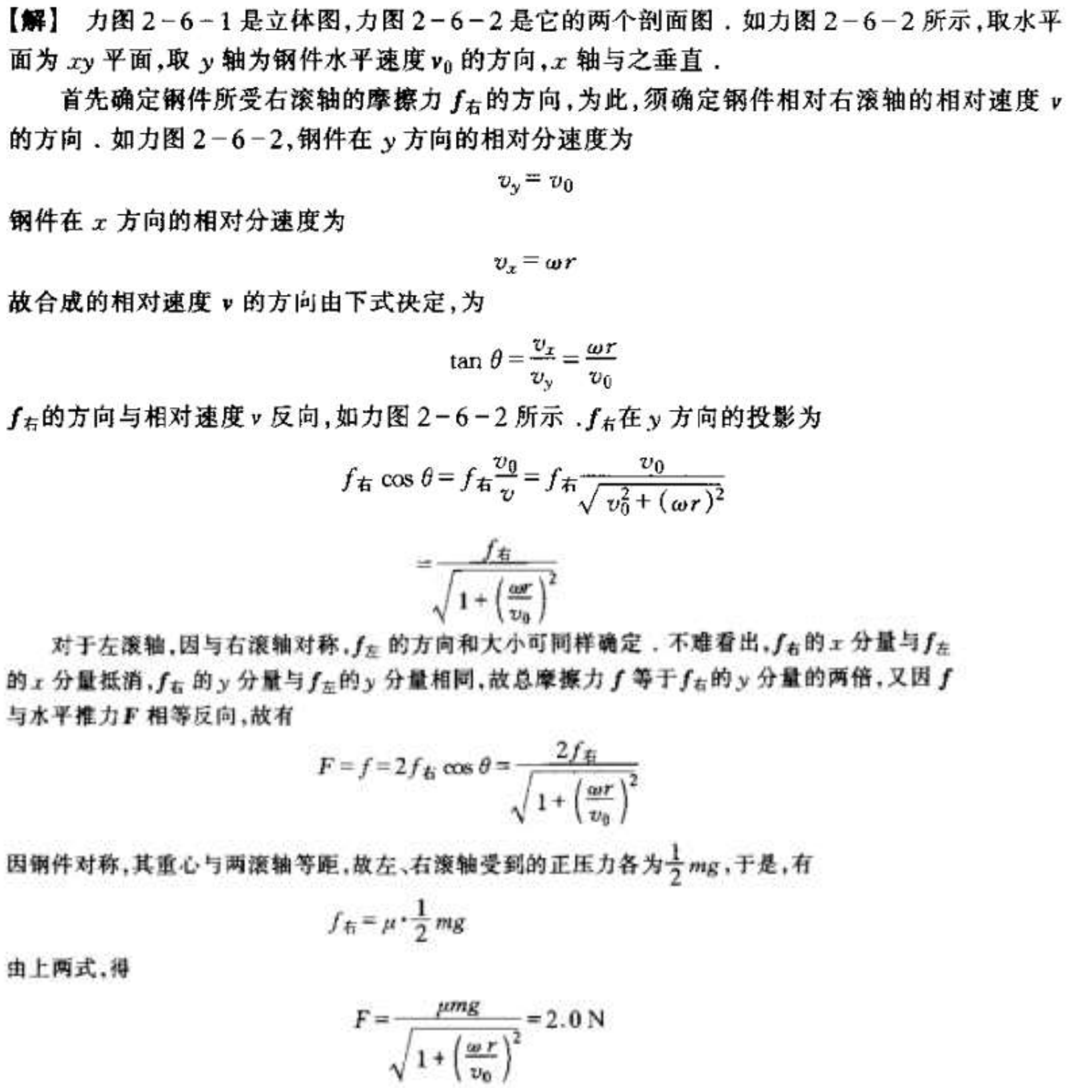
力图 2－6－1

相对运动：
1：读题，画大概的示意图
2：找参考系/参考点，一般选取不动的/与运动点距离一样（瞬心）/或者运动轨迹已知/或者你觉得你喜欢的
3：写相对运动关系式，如果你觉得不熟悉不确定，参考下面书写方式：

$$v_{a\to c}=v_{a\to b}+v_{b\to c}$$

这个式子可以帮助你理解相对运动，一般c点为地或者绝对不动点

Solution:



【解】 力图 2－6－1 是立体图，力图 2－6－2 是它的两个剖面图．如力图 2－6－2 所示，取水平面为 xy 平面，取 y 轴为钢件水平速度 $\boldsymbol{v}_0$ 的方向，x 轴与之垂直．

首先确定钢件所受右滚轴的摩擦力 $\boldsymbol{f}_{\text{右}}$ 的方向，为此，须确定钢件相对右滚轴的相对速度 $\boldsymbol{v}$ 的方向．如力图 2－6－2，钢件在 y 方向的相对分速度为

$$v_y = v_0$$

钢件在 x 方向的相对分速度为

$$v_x = \omega r$$

故合成的相对速度 $\boldsymbol{v}$ 的方向由下式决定，为

$$\tan\theta = \frac{v_x}{v_y} = \frac{\omega r}{v_0}$$

$\boldsymbol{f}_{\text{右}}$ 的方向与相对速度 v 反向，如力图 2－6－2 所示．$\boldsymbol{f}_{\text{右}}$ 在 y 方向的投影为

$$f_{\text{右}}\cos\theta = f_{\text{右}}\frac{v_0}{v} = f_{\text{右}}\frac{v_0}{\sqrt{v_0^2+(\omega r)^2}}$$

$$= \frac{f_{\text{右}}}{\sqrt{1+\left(\frac{\omega r}{v_0}\right)^2}}$$

对于左滚轴，因与右滚轴对称，$\boldsymbol{f}_{\text{左}}$ 的方向和大小可同样确定．不难看出，$\boldsymbol{f}_{\text{右}}$ 的 x 分量与 $\boldsymbol{f}_{\text{左}}$ 的 x 分量抵消，$\boldsymbol{f}_{\text{右}}$ 的 y 分量与 $\boldsymbol{f}_{\text{左}}$ 的 y 分量相同，故总摩擦力 $\boldsymbol{f}$ 等于 $\boldsymbol{f}_{\text{右}}$ 的 y 分量的两倍，又因 $\boldsymbol{f}$ 与水平推力 $\boldsymbol{F}$ 相等反向，故有

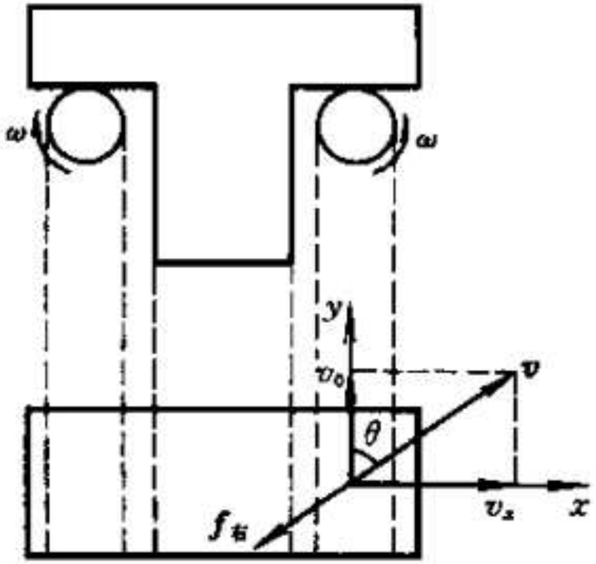
$$F = f = 2f_{\text{右}}\cos\theta = \frac{2f_{\text{右}}}{\sqrt{1+\left(\frac{\omega r}{v_0}\right)^2}}$$

因钢件对称，其重心与两滚轴等距，故左、右滚轴受到的正压力各为 $\frac{1}{2}mg$，于是，有

$$f_{\text{右}} = \mu\cdot\frac{1}{2}mg$$

由上两式，得

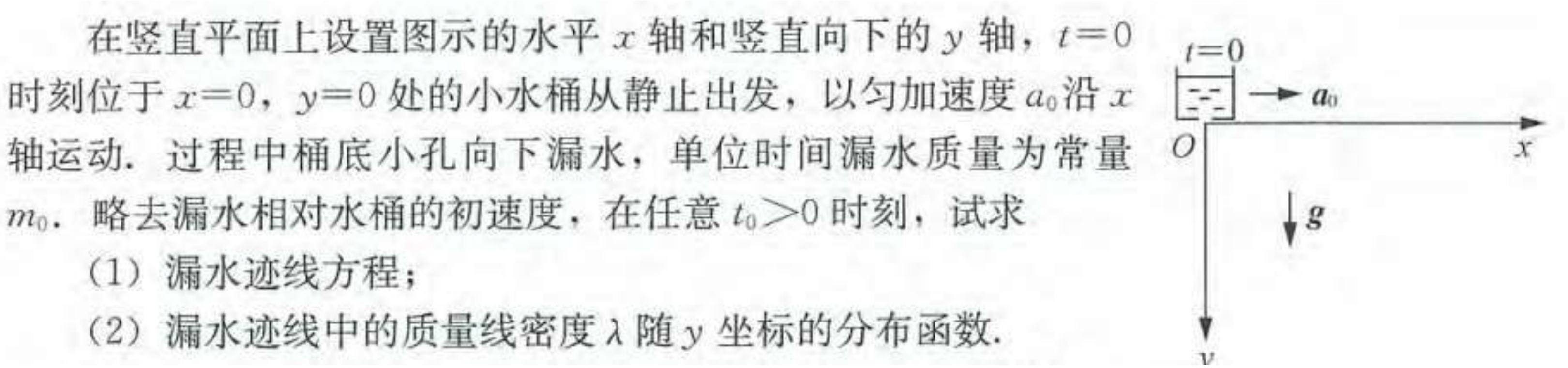
$$F = \frac{\mu mg}{\sqrt{1+\left(\frac{\omega r}{v_0}\right)^2}} = 2.0\ \text{N}$$



力图 2－6－2

Trajectory(almost without any physics knowledge)

You just need your hands



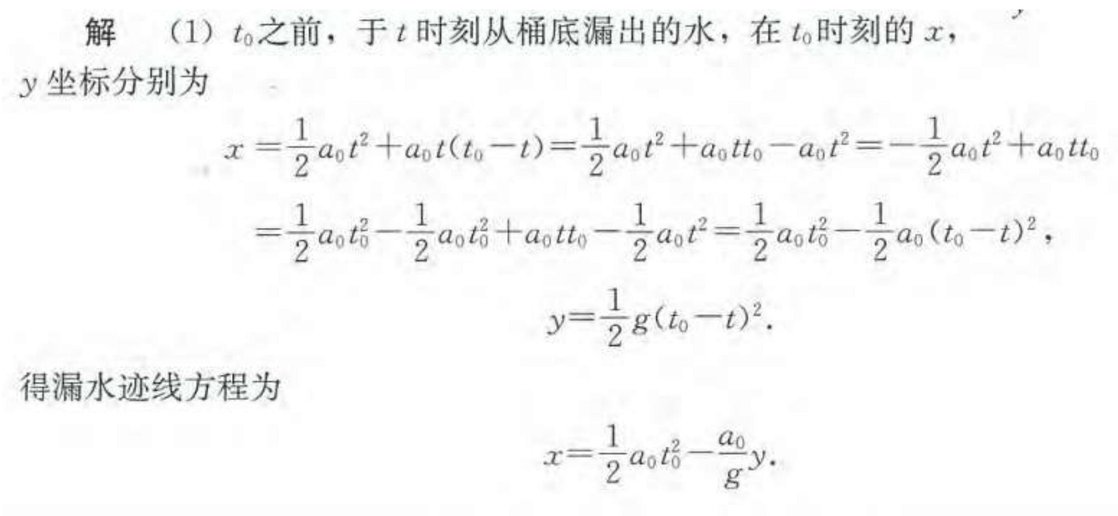
在竖直平面上设置图示的水平 x 轴和竖直向下的 y 轴，$t=0$ 时刻位于 $x=0$，$y=0$ 处的小水桶从静止出发，以匀加速度 a_0 沿 x 轴运动．过程中桶底小孔向下漏水，单位时间漏水质量为常量 m_0．略去漏水相对水桶的初速度，在任意 $t_0>0$ 时刻，试求

（1）漏水迹线方程；

（2）漏水迹线中的质量线密度 λ 随 y 坐标的分布函数．

轨迹问题：
1：寻找坐标系，看有没有明显的不变量或者几何关系
如果存在定长“摆”长/定角度或者摆长与角度的关系已知，选取极坐标
如果告知运动过程中切向和法向上物理量的关系，选取自然坐标
如果什么都不明显或者计算技巧强，直接直角坐标
2：列坐标系里的运动方程，观察
3：取t→t+Δt 或者解简单微分方程(见rc2-math part)，解运动学微分方程组时常用洛必达或者两个式子（x，v，a）形式相近，可以发现比例或者线性关系

Solution:



解　（1）t_0之前，于 t 时刻从桶底漏出的水，在 t_0时刻的 x，y 坐标分别为

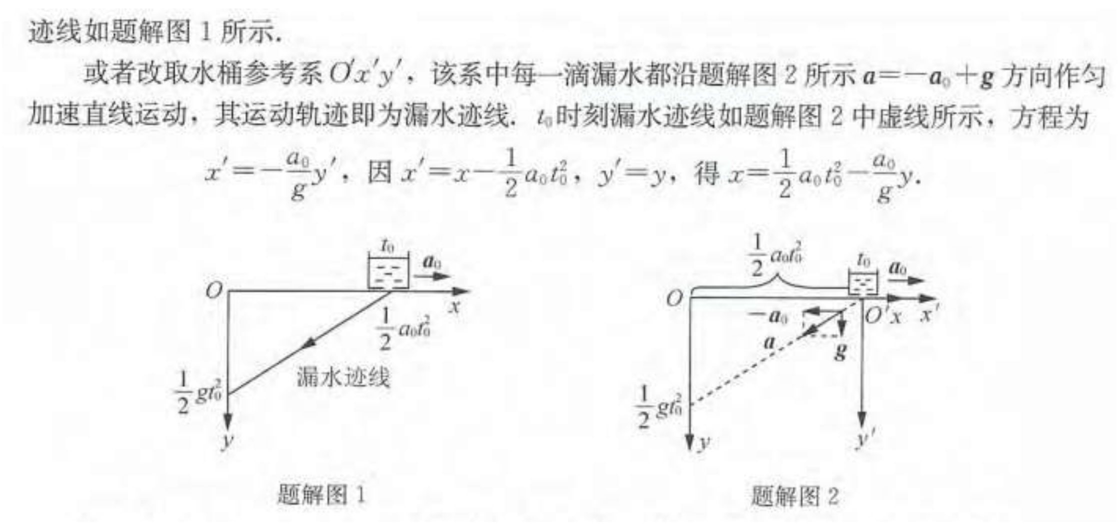
$$x=\frac{1}{2}a_0t^2+a_0t(t_0-t)=\frac{1}{2}a_0t^2+a_0tt_0-a_0t^2=-\frac{1}{2}a_0t^2+a_0tt_0$$

$$=\frac{1}{2}a_0t_0^2-\frac{1}{2}a_0t_0^2+a_0tt_0-\frac{1}{2}a_0t^2=\frac{1}{2}a_0t_0^2-\frac{1}{2}a_0(t_0-t)^2,$$

$$y=\frac{1}{2}g(t_0-t)^2.$$

得漏水迹线方程为

$$x=\frac{1}{2}a_0t_0^2-\frac{a_0}{g}y.$$



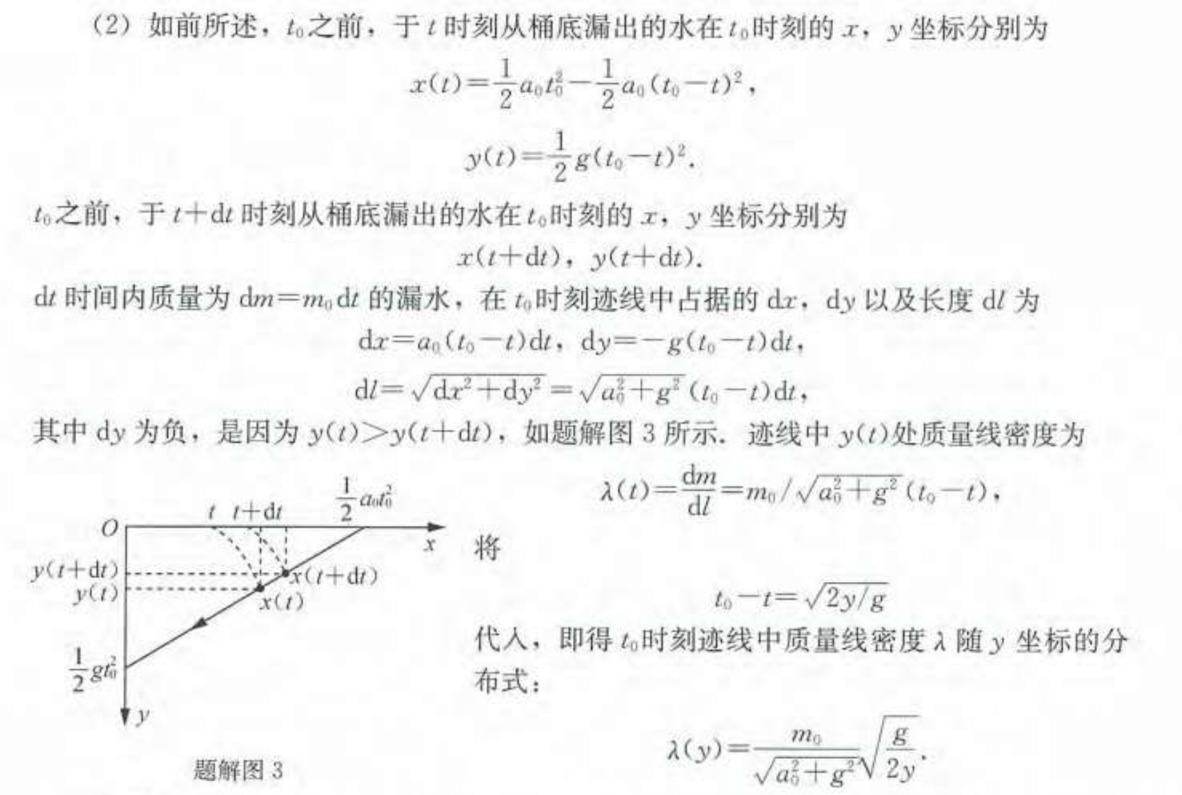
迹线如题解图 1 所示.

或者改取水桶参考系 $O'x'y'$，该系中每一滴漏水都沿题解图 2 所示 $\boldsymbol{a}=-\boldsymbol{a}_0+\boldsymbol{g}$ 方向作匀加速直线运动，其运动轨迹即为漏水迹线. t_0时刻漏水迹线如题解图 2 中虚线所示，方程为

$$x'=-\frac{a_0}{g}y'，因\ x'=x-\frac{1}{2}a_0t_0^2，\ y'=y，得\ x=\frac{1}{2}a_0t_0^2-\frac{a_0}{g}y.$$

题解图 1

题解图 2



（2）如前所述，t_0之前，于 t 时刻从桶底漏出的水在 t_0时刻的 x，y 坐标分别为

$$x(t)=\frac{1}{2}a_0t_0^2-\frac{1}{2}a_0(t_0-t)^2,$$

$$y(t)=\frac{1}{2}g(t_0-t)^2.$$

t_0之前，于 $t+\mathrm{d}t$ 时刻从桶底漏出的水在 t_0时刻的 x，y 坐标分别为

$$x(t+\mathrm{d}t)，\ y(t+\mathrm{d}t).$$

$\mathrm{d}t$ 时间内质量为 $\mathrm{d}m=m_0\mathrm{d}t$ 的漏水，在 t_0时刻迹线中占据的 $\mathrm{d}x$，$\mathrm{d}y$ 以及长度 $\mathrm{d}l$ 为

$$\mathrm{d}x=a_0(t_0-t)\mathrm{d}t，\ \mathrm{d}y=-g(t_0-t)\mathrm{d}t,$$

$$\mathrm{d}l=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}=\sqrt{a_0^2+g^2}(t_0-t)\mathrm{d}t,$$

其中 $\mathrm{d}y$ 为负，是因为 $y(t)>y(t+\mathrm{d}t)$，如题解图 3 所示. 迹线中 $y(t)$处质量线密度为

$$\lambda(t)=\frac{\mathrm{d}m}{\mathrm{d}l}=m_0/\sqrt{a_0^2+g^2}(t_0-t),$$

将

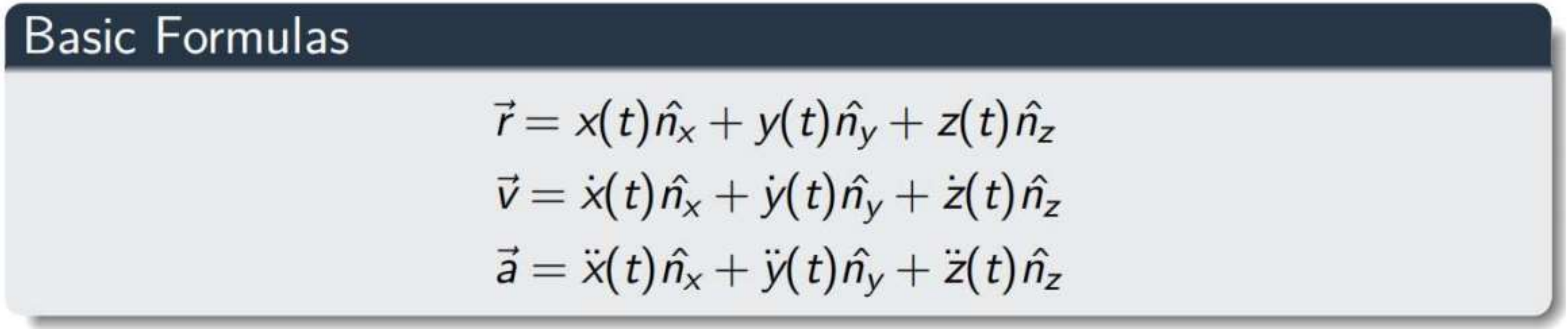
$$t_0-t=\sqrt{2y/g}$$

代入，即得 t_0时刻迹线中质量线密度 λ 随 y 坐标的分布式：

$$\lambda(y)=\frac{m_0}{\sqrt{a_0^2+g^2}}\sqrt{\frac{g}{2y}}.$$

题解图 3

Cartesian Coordinate system



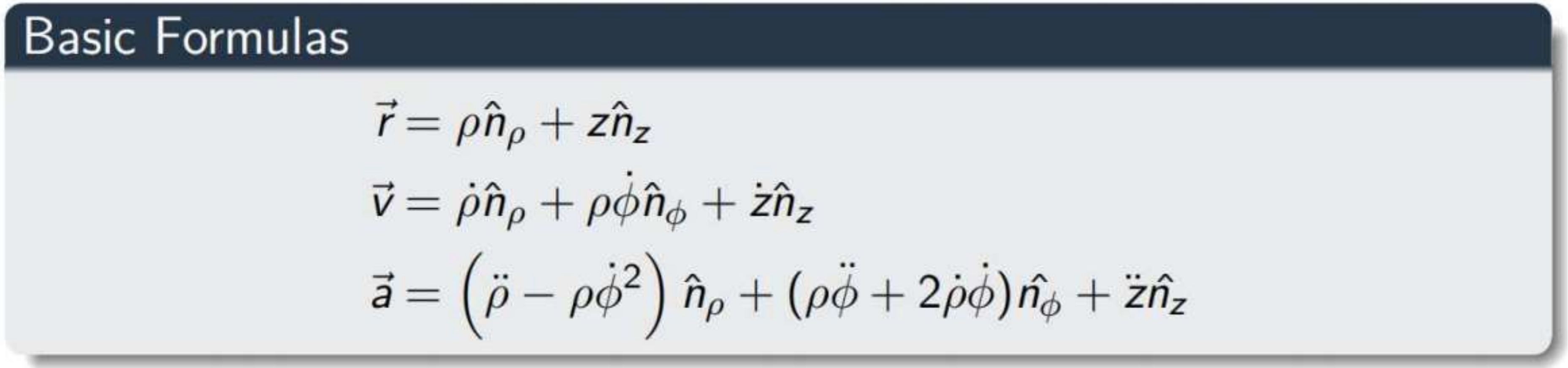
Basic Formulas

$$\vec{r} = x(t)\hat{n}_x + y(t)\hat{n}_y + z(t)\hat{n}_z$$

$$\vec{v} = \dot{x}(t)\hat{n}_x + \dot{y}(t)\hat{n}_y + \dot{z}(t)\hat{n}_z$$

$$\vec{a} = \ddot{x}(t)\hat{n}_x + \ddot{y}(t)\hat{n}_y + \ddot{z}(t)\hat{n}_z$$

Cylindrical Coordinate system



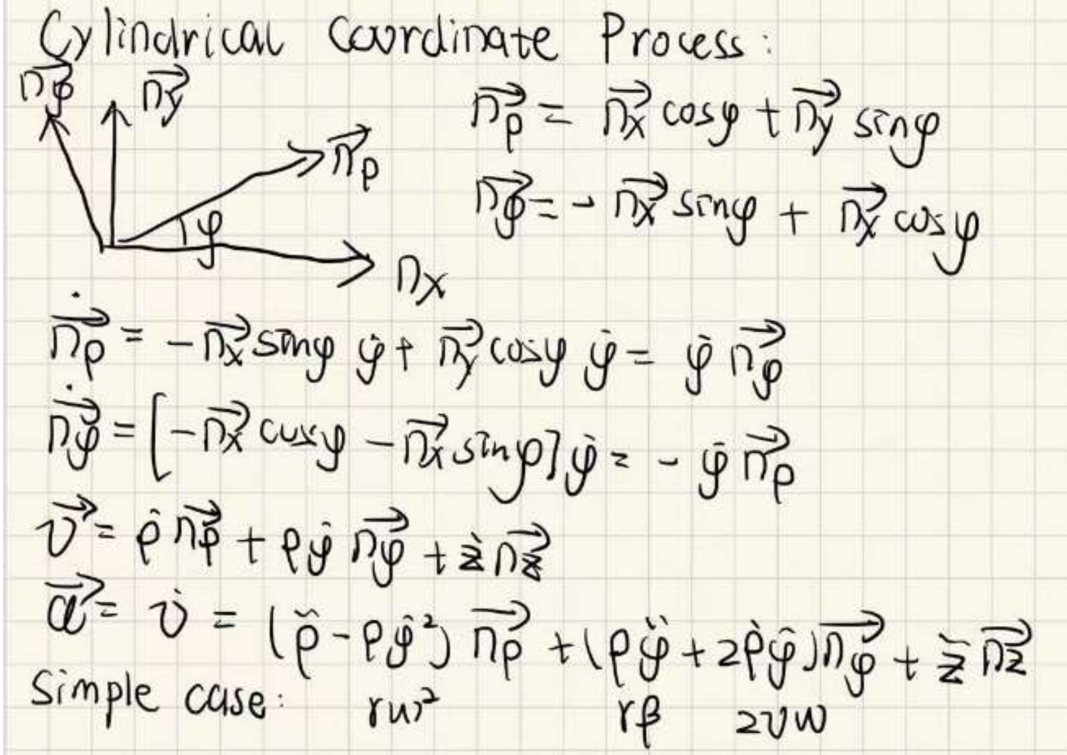
Basic Formulas

$$\vec{r} = \rho\hat{n}_\rho + z\hat{n}_z$$

$$\vec{v} = \dot{\rho}\hat{n}_\rho + \rho\dot{\phi}\hat{n}_\phi + \dot{z}\hat{n}_z$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{n}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{n}_\phi + \ddot{z}\hat{n}_z$$

Cylindrical Coordinate process



Cylindrical Coordinate Process:

$\vec{n_\rho} = \vec{n_x}\cos\varphi + \vec{n_y}\sin\varphi$

$\vec{n_\varphi} = -\vec{n_x}\sin\varphi + \vec{n_x}\cos\varphi$

$\dot{\vec{n_\rho}} = -\vec{n_x}\sin\varphi\,\dot\varphi + \vec{n_y}\cos\varphi\,\dot\varphi = \dot\varphi\,\vec{n_\varphi}$

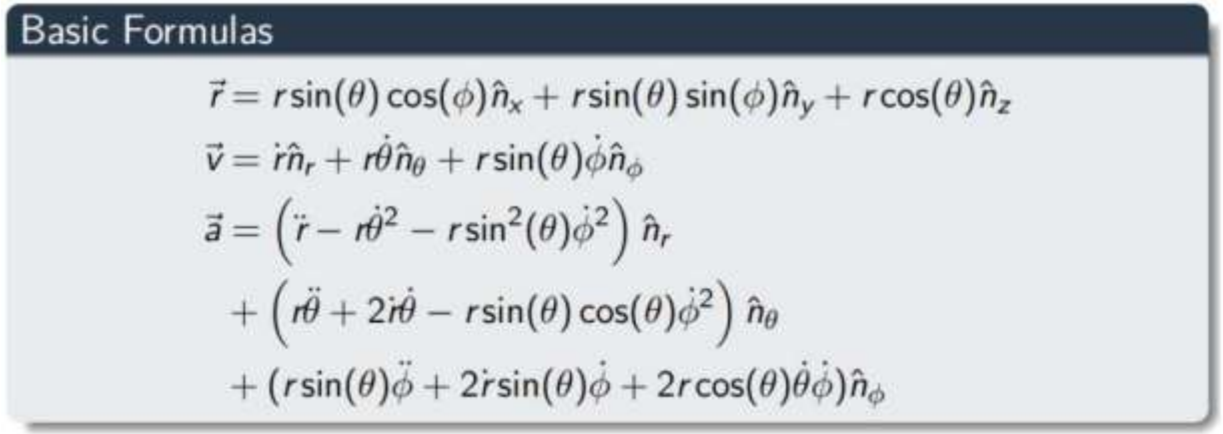
$\dot{\vec{n_\varphi}} = [-\vec{n_x}\cos\varphi - \vec{n_x}\sin\varphi]\dot\varphi = -\dot\varphi\,\vec{n_\rho}$

$\vec{v} = \dot\rho\,\vec{n_\rho} + \rho\dot\varphi\,\vec{n_\varphi} + \dot z\,\vec{n_z}$

$\vec{a} = \dot v = (\ddot\rho - \rho\dot\varphi^2)\,\vec{n_\rho} + (\rho\ddot\varphi + 2\dot\rho\dot\varphi)\,\vec{n_\varphi} + \ddot z\,\vec{n_z}$

Simple case: $r\omega^2$ $\quad r\beta \quad 2v\omega$

Spherical Coordinate system



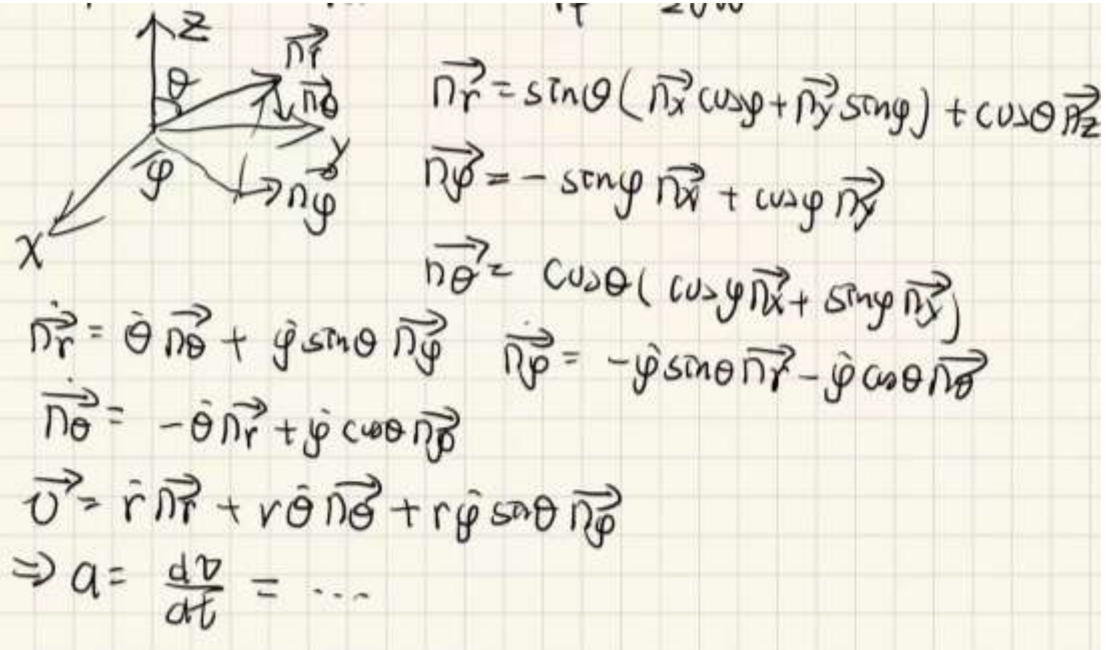
Basic Formulas

$$\vec{r} = r\sin(\theta)\cos(\phi)\hat{n}_x + r\sin(\theta)\sin(\phi)\hat{n}_y + r\cos(\theta)\hat{n}_z$$

$$\vec{v} = \dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta + r\sin(\theta)\dot{\phi}\hat{n}_\phi$$

$$\begin{aligned}\vec{a} = &\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2(\theta)\dot{\phi}^2\right)\hat{n}_r\\ &+ \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin(\theta)\cos(\theta)\dot{\phi}^2\right)\hat{n}_\theta\\ &+ (r\sin(\theta)\ddot{\phi} + 2\dot{r}\sin(\theta)\dot{\phi} + 2r\cos(\theta)\dot{\theta}\dot{\phi})\hat{n}_\phi\end{aligned}$$

Spherical Coordinate process



$$\vec{n_r} = \sin\theta(\vec{n_x}\cos\varphi + \vec{n_y}\sin\varphi) + \cos\theta\,\vec{n_z}$$

$$\vec{n_\varphi} = -\sin\varphi\,\vec{n_x} + \cos\varphi\,\vec{n_y}$$

$$\vec{n_\theta} = \cos\theta(\cos\varphi\,\vec{n_x} + \sin\varphi\,\vec{n_y})$$

$$\dot{\vec{n_r}} = \dot\theta\,\vec{n_\theta} + \dot\varphi\sin\theta\,\vec{n_\varphi} \qquad \dot{\vec{n_\varphi}} = -\dot\varphi\sin\theta\,\vec{n_r} - \dot\varphi\cos\theta\,\vec{n_\theta}$$

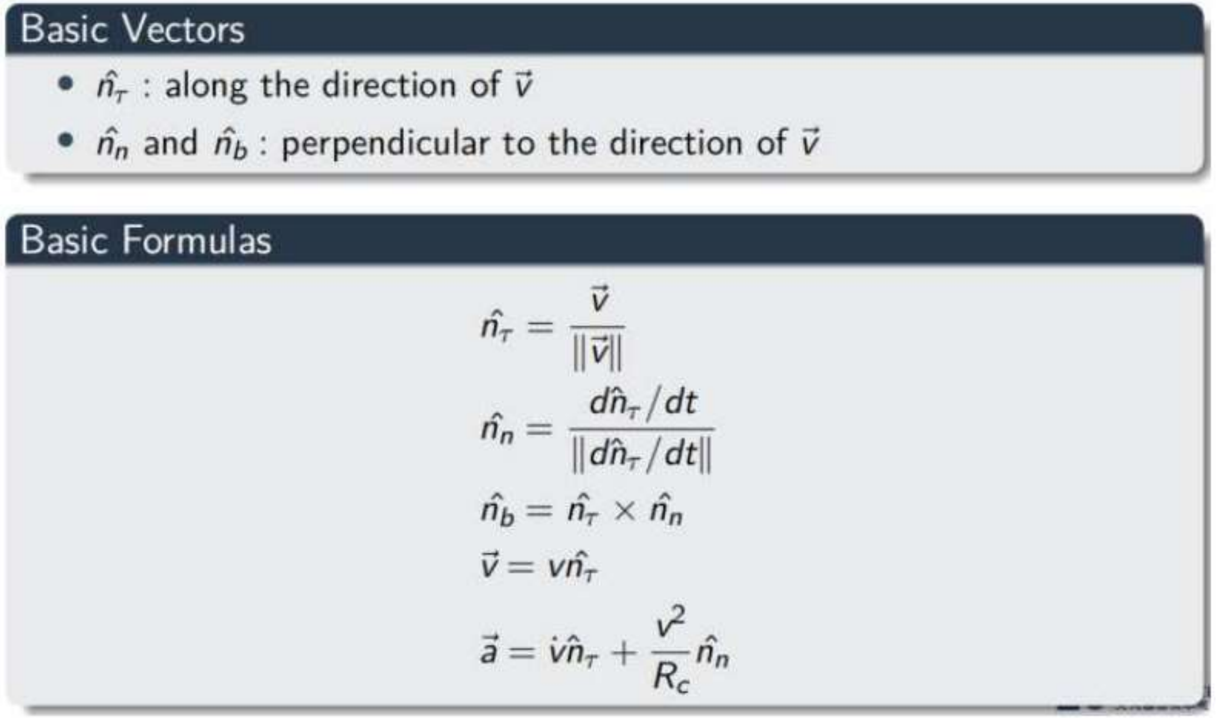
$$\dot{\vec{n_\theta}} = -\dot\theta\,\vec{n_r} + \dot\varphi\cos\theta\,\vec{n_\varphi}$$

$$\vec{v} = \dot r\,\vec{n_r} + r\dot\theta\,\vec{n_\theta} + r\dot\varphi\sin\theta\,\vec{n_\varphi}$$

$$\Rightarrow a = \frac{dv}{dt} = \cdots$$

常常与牛顿定理结合，变综合题

Natural Coordinate



Basic Vectors

- $\hat{n}_\tau$: along the direction of $\vec{v}$
- $\hat{n}_n$ and $\hat{n}_b$: perpendicular to the direction of $\vec{v}$

Basic Formulas

$$\hat{n}_\tau = \frac{\vec{v}}{\|\vec{v}\|}$$

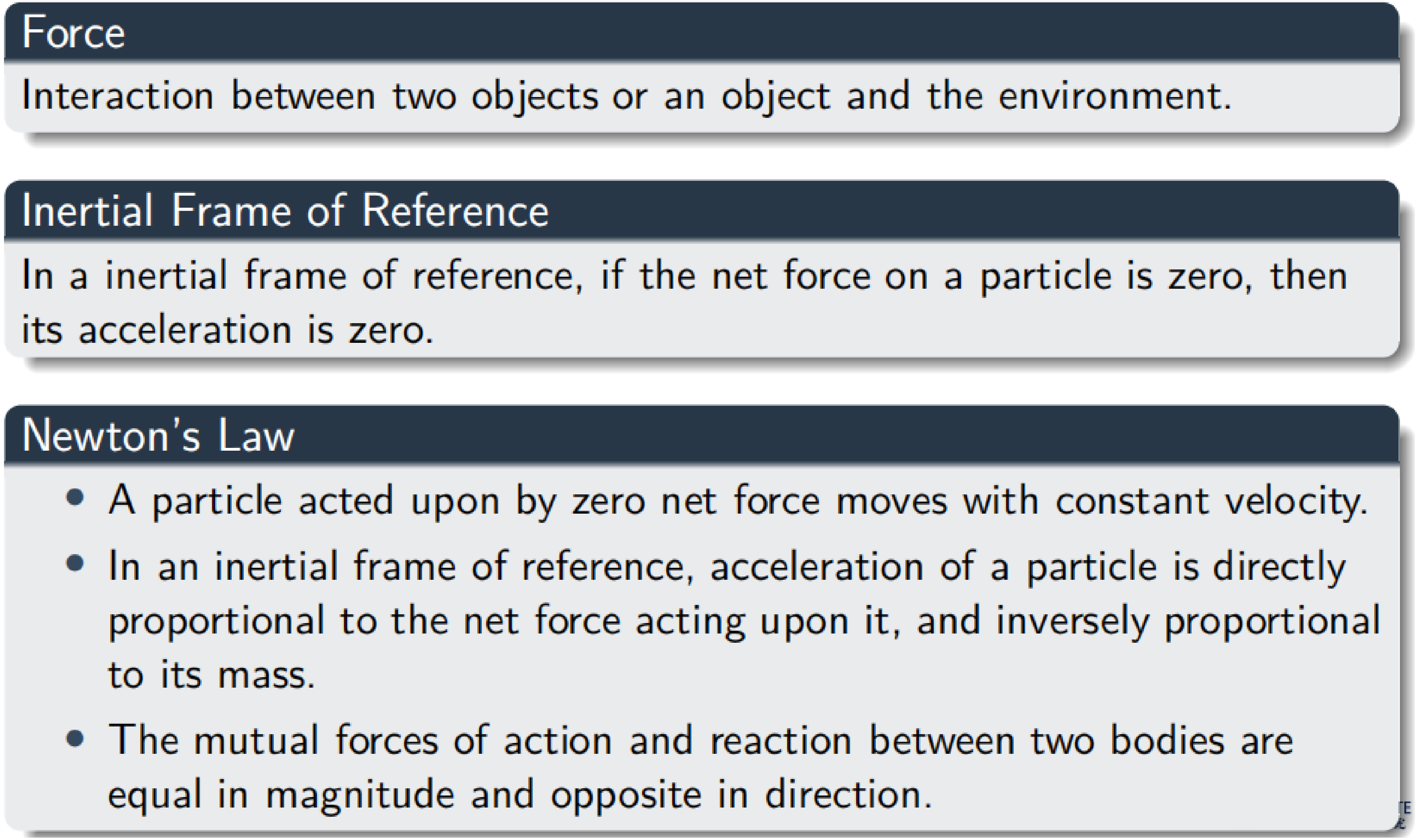
$$\hat{n}_n = \frac{d\hat{n}_\tau/dt}{\|d\hat{n}_\tau/dt\|}$$

$$\hat{n}_b = \hat{n}_\tau \times \hat{n}_n$$

$$\vec{v} = v\hat{n}_\tau$$

$$\vec{a} = \dot{v}\hat{n}_\tau + \frac{v^2}{R_c}\hat{n}_n$$

Force and Inertial Frame of Reference



Force

Interaction between two objects or an object and the environment.

Inertial Frame of Reference

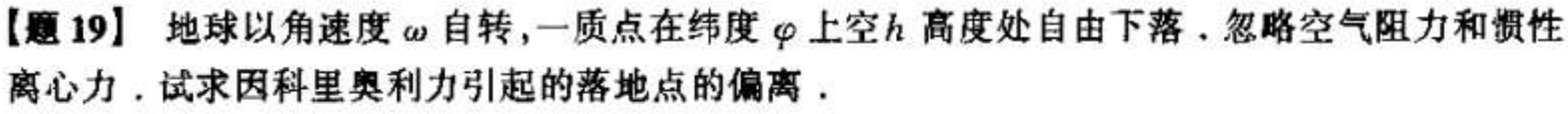
In a inertial frame of reference, if the net force on a particle is zero, then its acceleration is zero.

Newton's Law

- A particle acted upon by zero net force moves with constant velocity.
- In an inertial frame of reference, acceleration of a particle is directly proportional to the net force acting upon it, and inversely proportional to its mass.
- The mutual forces of action and reaction between two bodies are equal in magnitude and opposite in direction.

牛一牛二牛三考试的默写题部分应该会考察（确信）

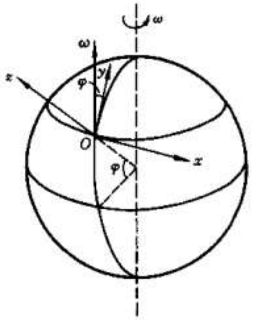
Similar but still difficult problems:



【题 19】 地球以角速度 ω 自转，一质点在纬度 φ 上空 h 高度处自由下落．忽略空气阻力和惯性离心力．试求因科里奥利力引起的落地点的偏离．

这题不想讲了，但是可以提供大概的思路
1：列动力方程 $F = ma$
2：（看限制条件忽略一些加速度）
3：解运动方程

Solution:



力图 2-19-1

【解】 如图，取地球为参考系，取直角坐标 $Oxyz$，原点 O 在落体初始位置正下方的地面上，x 轴与纬度线相切指向东方，y 轴与经度线相切指向北方，z 轴垂直地面向上．图中 $\boldsymbol{\omega}$ 为地球自转角速度，质点初始位置为(0,0,h)，初速度为(0,0,0).

质点受重力 $m\boldsymbol{g}$ 及科里奥利力 $-2m\boldsymbol{\omega}\times\boldsymbol{v}$，由牛顿第二定律，其运动方程为

$$m\ddot{\boldsymbol{r}}=m\boldsymbol{g}-2m\boldsymbol{\omega}\times\boldsymbol{v} \qquad (1)$$

式中 $\boldsymbol{r}$，$\boldsymbol{v}$，$\ddot{\boldsymbol{r}}$ 分别是质点的位矢，速度和加速度．式中有关量的具体形式为

$$\ddot{\boldsymbol{r}}=(\ddot{x},\ddot{y},\ddot{z})$$
$$\boldsymbol{g}=(0,0,-g)$$
$$\boldsymbol{\omega}=(0,\omega\cos\varphi,\omega\sin\varphi)$$
$$\boldsymbol{v}=(\dot{x},\dot{y},\dot{z})$$

$$\boldsymbol{\omega}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & \omega\cos\varphi & \omega\sin\varphi\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix}$$
$$=(\omega\cos\varphi\cdot\dot{z}-\omega\sin\varphi\cdot\dot{y})\boldsymbol{i}+\omega\sin\varphi\cdot\dot{x}\boldsymbol{j}-\omega\cos\varphi\cdot\dot{x}\boldsymbol{k}$$

其中 $\boldsymbol{i}$，$\boldsymbol{j}$，$\boldsymbol{k}$ 是 x，y，z 轴的单位矢量．于是，(1)式的分量形式为

$$\begin{cases}\ddot{x}=-2\omega\left(\cos\varphi\dfrac{\mathrm{d}z}{\mathrm{d}t}-\sin\varphi\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)\\ \ddot{y}=-2\omega\sin\varphi\dfrac{\mathrm{d}x}{\mathrm{d}t}\\ \ddot{z}=-g+2\omega\cos\varphi\dfrac{\mathrm{d}x}{\mathrm{d}t}\end{cases} \qquad (2)$$

或

$$\begin{cases}\mathrm{d}\dot{x}=-2\omega(\cos\varphi\mathrm{d}z-\sin\varphi\mathrm{d}y) & (3)\\ \mathrm{d}\dot{y}=-2\omega\sin\varphi\mathrm{d}x & (4)\\ \mathrm{d}\dot{z}=-g\mathrm{d}t+2\omega\cos\varphi\mathrm{d}x & (5)\end{cases}$$

(3)式积分，得

$$\dot{x}=-2\omega(z\cos\varphi-y\sin\varphi)+C$$

初条件为 $z=h$，$y=0$ 处(起始点)的 $\dot{x}=0$，故积分常量为

$$C=2\omega h\cos\varphi$$

代入，得

$$\dot{x}=-2\omega[(z-h)\cos\varphi-y\sin\varphi]$$

(4)式积分，得

$$\dot{y}=-2\omega x\sin\varphi+C$$

初条件为 $x=0$ 处，$\dot{y}=0$，故积分常量 $C=0$，代入，得

$$\dot{y}=-2\omega x\sin\varphi$$

(5)式积分，得

$$\dot{z}=-gt+2\omega x\cos\varphi+C$$

初条件为 $t=0$ 时，$x=0$，$\dot{z}=0$，故积分常量 $C=0$，代入，得

$$\dot{z}=-gt+2\omega x\cos\varphi$$

把上述由(3)、(4)、(5)式积分得出的 $\dot{x}$，$\dot{y}$，$\dot{z}$ 代入(2)式，得

$$\begin{cases}\ddot{x}=-2\omega(-gt\cos\varphi+2\omega x\cos^2\varphi+2\omega x\sin^2\varphi)\\ \quad=2\omega gt\cos\varphi-4\omega^2x\\ \ddot{y}=4\omega^2\sin\varphi[(z-h)\cos\varphi-y\sin\varphi]\\ \ddot{z}=-g-4\omega^2\cos\varphi[(z-h)\cos\varphi-y\sin\varphi]\end{cases}$$

因地球自转角速度 ω 较小，上式中各 ω^2 项可略，得

$$\begin{cases}\ddot{x}=2\omega gt\cos\varphi\\ \ddot{z}=-g\end{cases}$$

积分，得

$$\begin{cases}\dot{x}=(\omega g\cos\varphi)t^2+C_1\\ \dot{z}=-gt+C_2\end{cases}$$

初条件为 $t=0$ 时，$\dot{x}=0$，$\dot{z}=0$，故积分常量 $C_1=C_2=0$，代入，得

$$\begin{cases}\dot{x}=(\omega g\cos\varphi)t^2\\ \dot{z}=-gt\end{cases}$$

再积分，得

$$\begin{cases}x=\left(\dfrac{1}{3}\omega g\cos\varphi\right)t^3+C_3\\ z=-\dfrac{1}{2}gt^2+C_4\end{cases}$$

初条件为 $t=0$ 时，$x=0$，$z=h$，故积分常量 $C_3=0$，$C_4=h$，代入，得

$$\begin{cases}x=\left(\dfrac{1}{3}\omega g\cos\varphi\right)t^3\\ z=h-\dfrac{1}{2}gt^2\end{cases}$$

前已由(4)式积分得出 $\dot{y}=-2\omega x\sin\varphi$，把上述 x 代入，得

$$\dot{y}=\left(-\frac{2}{3}\omega^2g\sin\varphi\cos\varphi\right)t^3$$

积分，得

$$y=\left(-\frac{1}{6}\omega^2g\sin\varphi\cos\varphi\right)t^4+C$$

初条件为 $t=0$ 时，$y=0$，故积分常量 $C=0$，代入，得

$$y=\left(-\frac{1}{6}\omega^2g\sin\varphi\cos\varphi\right)t^4 \qquad (8)$$

(6)、(7)、(8)式就是质点自由下落时，位置随时间变化的规律．因落地点的 $z=0$，由(7)式，质点自高 h 处落地所需的时间为

$$t=\sqrt{\frac{2h}{g}}$$

代入(6)、(8)式，得出质点落地点的 x，y 坐标为

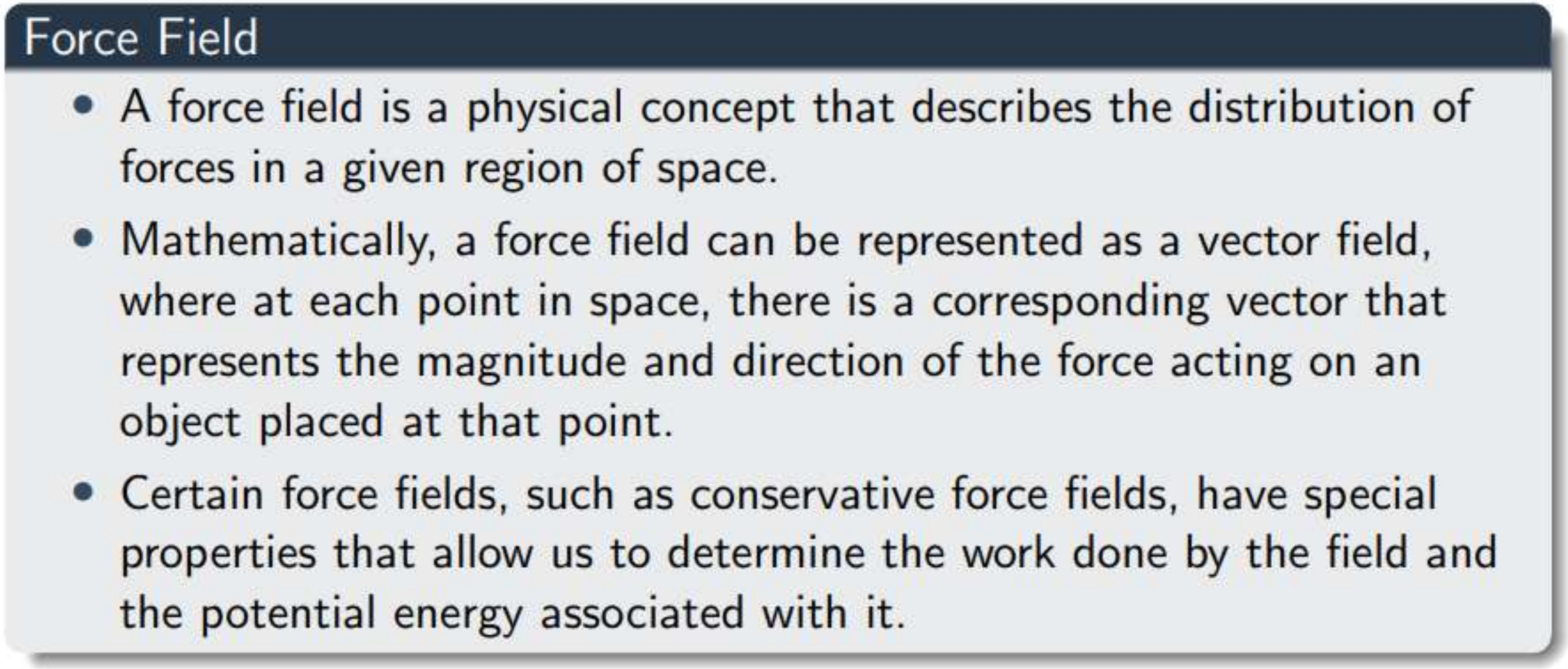
$$\begin{cases}x=\dfrac{1}{3}\omega g\cos\varphi\sqrt{\dfrac{8h^3}{g^3}}\\ \quad=\dfrac{1}{3}\sqrt{\dfrac{8h^3}{g}}\omega\cos\varphi\\ y=-\dfrac{2\omega^2h^2}{3g}\sin\varphi\cos\varphi\end{cases}$$

若质点在北纬 40°从离地面 100 m 的高度处自由下落，将 $\varphi-40^\circ$，$h-100\text{ m}$，$\omega-\dfrac{2\pi}{24\times3600}=7.3\times10^{-5}\text{ rad/s}$ 代入，得

$$\begin{cases}x=1.68\times10^{-2}\text{ m}=16.8\text{ mm}\\ y=-8.86\times10^{-5}\text{ m}=-0.0886\text{ mm}\end{cases}$$

即落地点向东偏 16.8 mm，向南偏 0.0886 mm(与东偏相比可忽略)，这正是科里奥利力的效应．

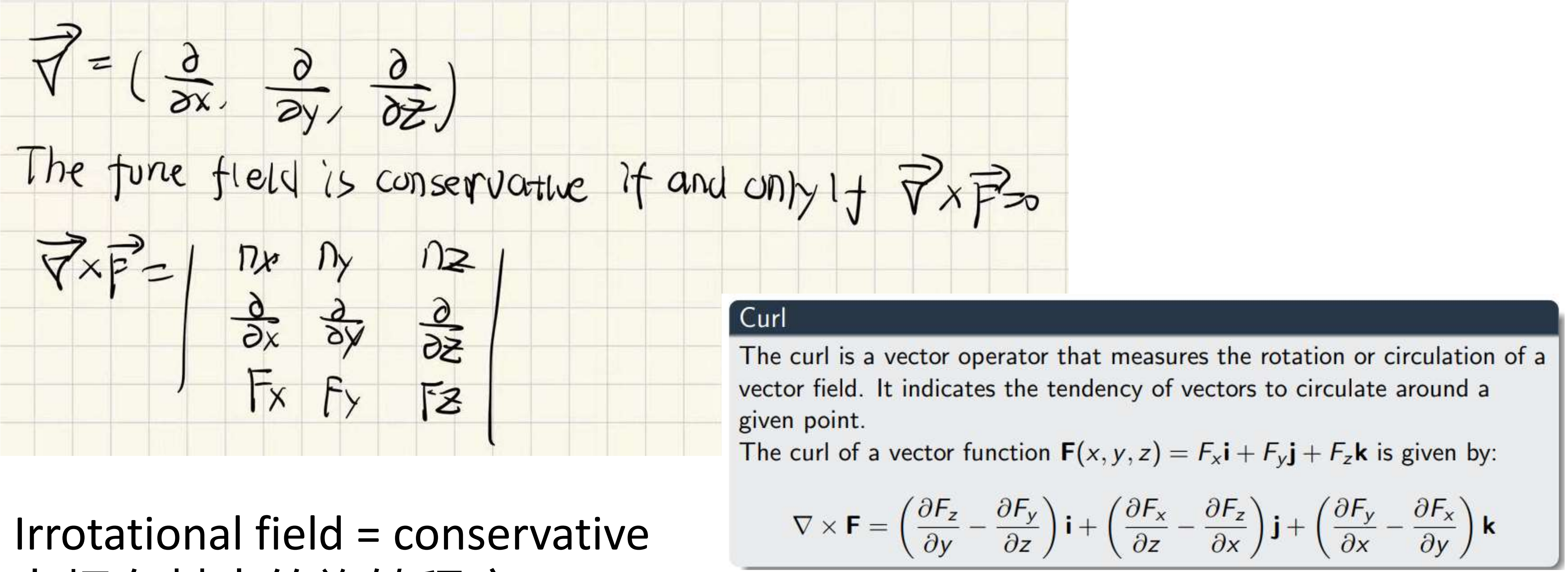
Force Field



Force Field

- A force field is a physical concept that describes the distribution of forces in a given region of space.
- Mathematically, a force field can be represented as a vector field, where at each point in space, there is a corresponding vector that represents the magnitude and direction of the force acting on an object placed at that point.
- Certain force fields, such as conservative force fields, have special properties that allow us to determine the work done by the field and the potential energy associated with it.

How to check if a force field is conservative



$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

The force field is conservative if and only if $\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} n_x & n_y & n_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Curl

The curl is a vector operator that measures the rotation or circulation of a vector field. It indicates the tendency of vectors to circulate around a given point.

The curl of a vector function $\mathbf{F}(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ is given by:

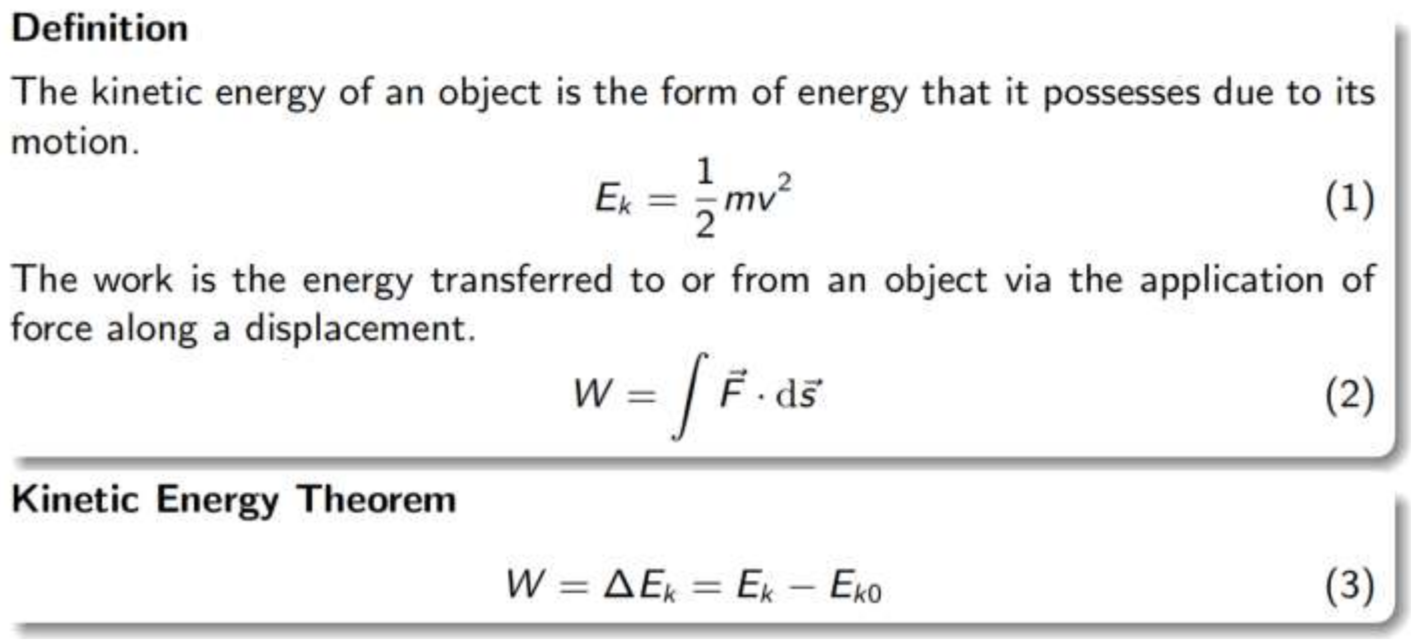
$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}$$

Irrotational field = conservative

力场在某点的旋转程度

旋度的大小是矢量场在某点周围的旋转强度

Kinetic Energy



Definition

The kinetic energy of an object is the form of energy that it possesses due to its motion.

$$E_k = \frac{1}{2}mv^2 \tag{1}$$

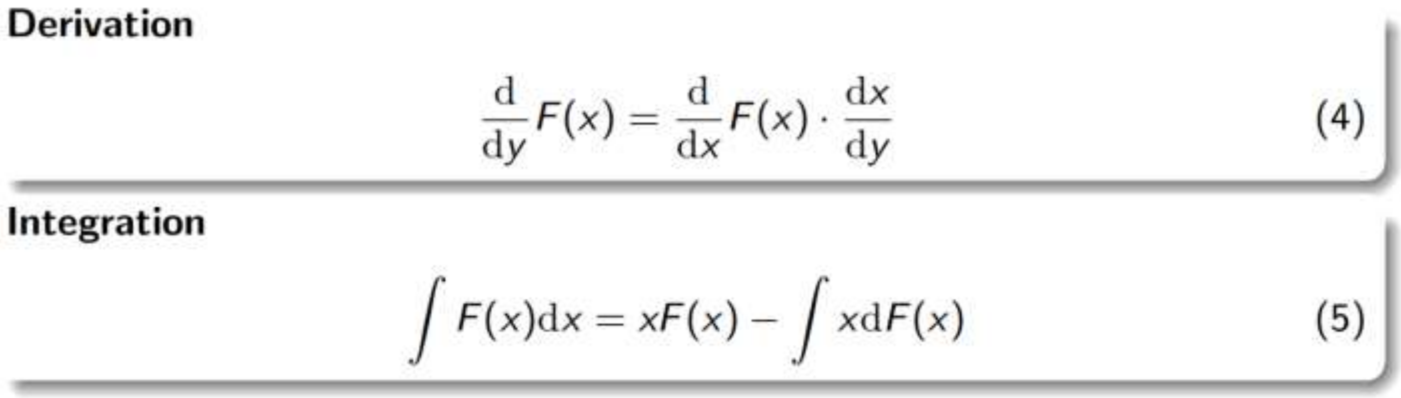
The work is the energy transferred to or from an object via the application of force along a displacement.

$$W = \int \vec{F} \cdot \mathrm{d}\vec{s} \tag{2}$$

Kinetic Energy Theorem

$$W = \Delta E_k = E_k - E_{k0} \tag{3}$$

Maybe used rule



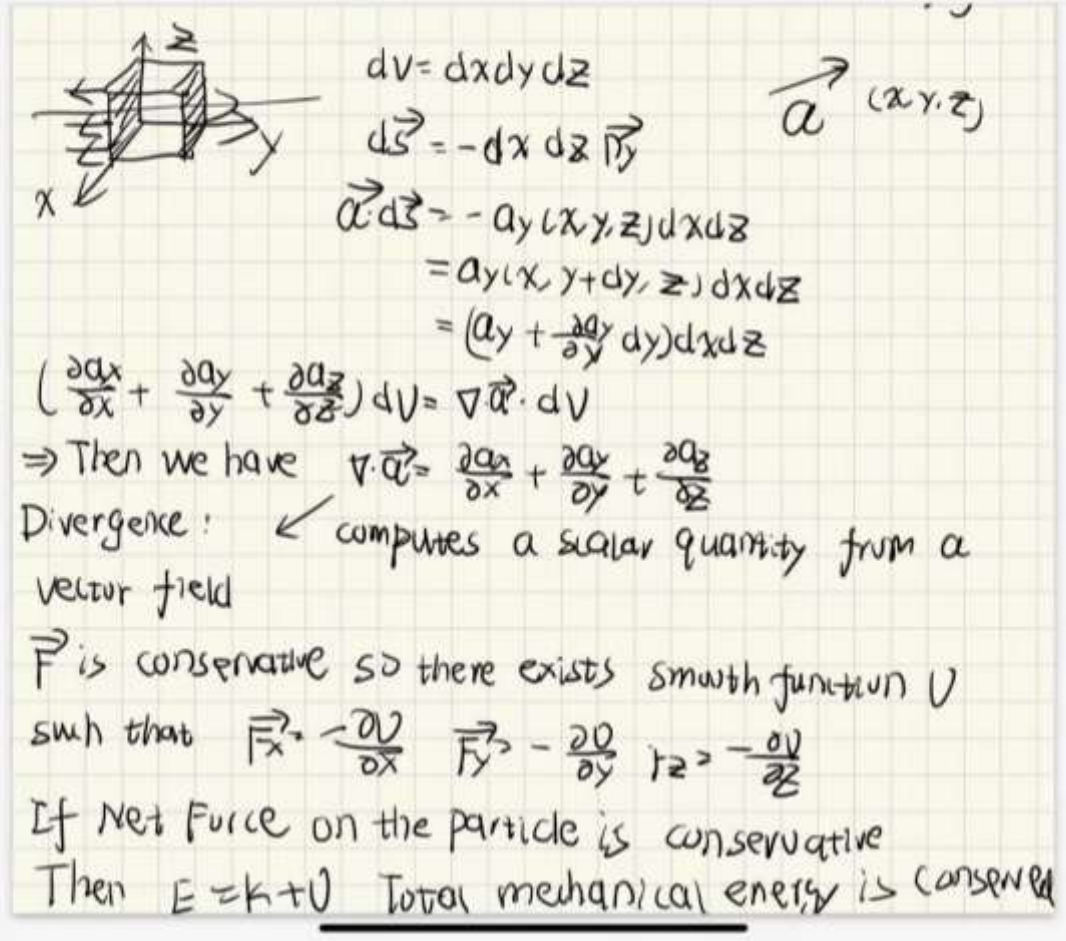
Derivation

$$\frac{\mathrm{d}}{\mathrm{d}y}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x) \cdot \frac{\mathrm{d}x}{\mathrm{d}y} \tag{4}$$

Integration

$$\int F(x)\mathrm{d}x = xF(x) - \int \mathrm{x}\mathrm{d}F(x) \tag{5}$$

Divergence of Potential Energy



$dV = dx\,dy\,dz$

$\vec{a}\ (x, y, z)$

$d\vec{S} = -dx\,dz\,\vec{n_y}$

$\vec{a}\cdot d\vec{S} = -a_y(x, y, z)\,dx\,dz$

$= a_y(x, y+dy, z)\,dx\,dz$

$= (a_y + \frac{\partial a_y}{\partial y}dy)\,dx\,dz$

$(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z})\,dV = \nabla\vec{a}\cdot dV$

⇒ Then we have $\nabla\cdot\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

Divergence: ↙ computes a scalar quantity from a vector field

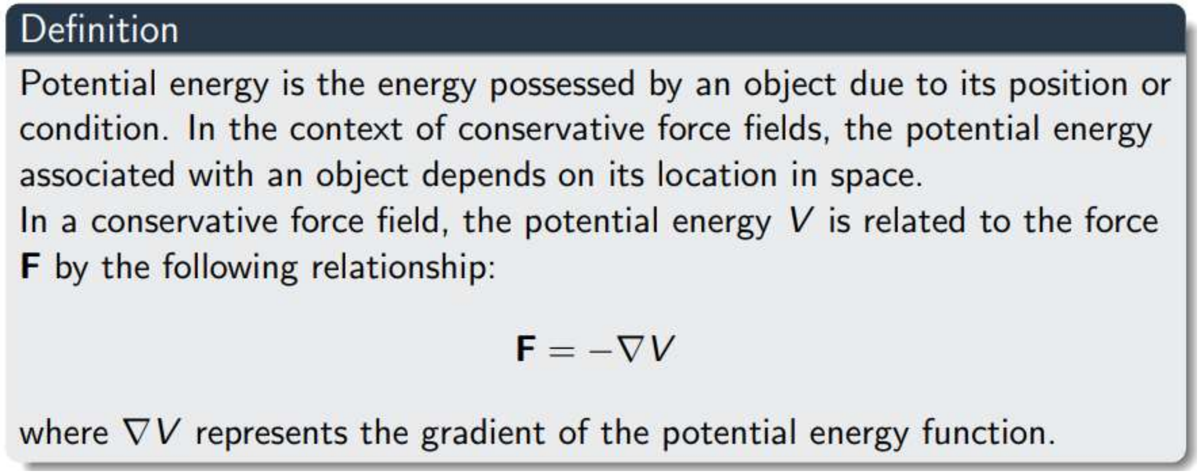
$\vec{F}$ is conservative so there exists smooth function U such that $\vec{F_x} = -\frac{\partial U}{\partial x}$ $\vec{F_y} = -\frac{\partial U}{\partial y}$ $F_z = -\frac{\partial U}{\partial z}$

If Net Force on the particle is conservative

Then $E = K + U$ Total mechanical energy is conserved

散度描述的是矢量场在某点的发散程度
散度为正表示矢量场在某点是源，有流出
散度为负表示矢量场在某点是汇，有流入

Potential Energy



Definition

Potential energy is the energy possessed by an object due to its position or condition. In the context of conservative force fields, the potential energy associated with an object depends on its location in space.

In a conservative force field, the potential energy V is related to the force $\mathbf{F}$ by the following relationship:

$$\mathbf{F} = -\nabla V$$

where ∇V represents the gradient of the potential energy function.

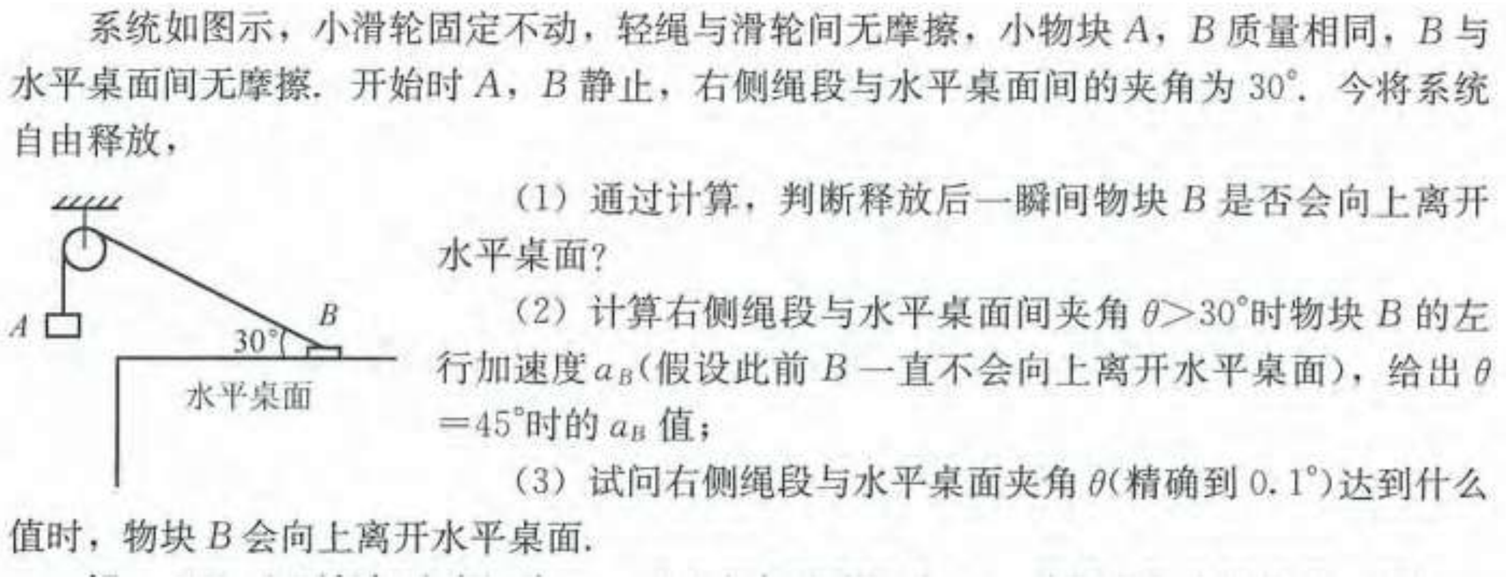
Key Concepts about Energy

功:
Work: measure the effort put into moving particle from one place to another
动能定理:
Work-KE Theorem: work done by net force on a particle is equal to the change in particle’s KE(Kinetic energy)

Key Concepts about Momentum

冲量定理:
The momentum-impulse theorem relates the change in momentum of an object to the impulse applied to it. It states that the impulse J acting on an object is equal to the change in momentum Δp of the object.
动量守恒:
Conservation of momentum states that the total momentum of an isolated system remains constant if no external forces are acting on it.

Simple Question



系统如图示，小滑轮固定不动，轻绳与滑轮间无摩擦，小物块 A，B 质量相同，B 与水平桌面间无摩擦．开始时 A，B 静止，右侧绳段与水平桌面间的夹角为 30°．今将系统自由释放，

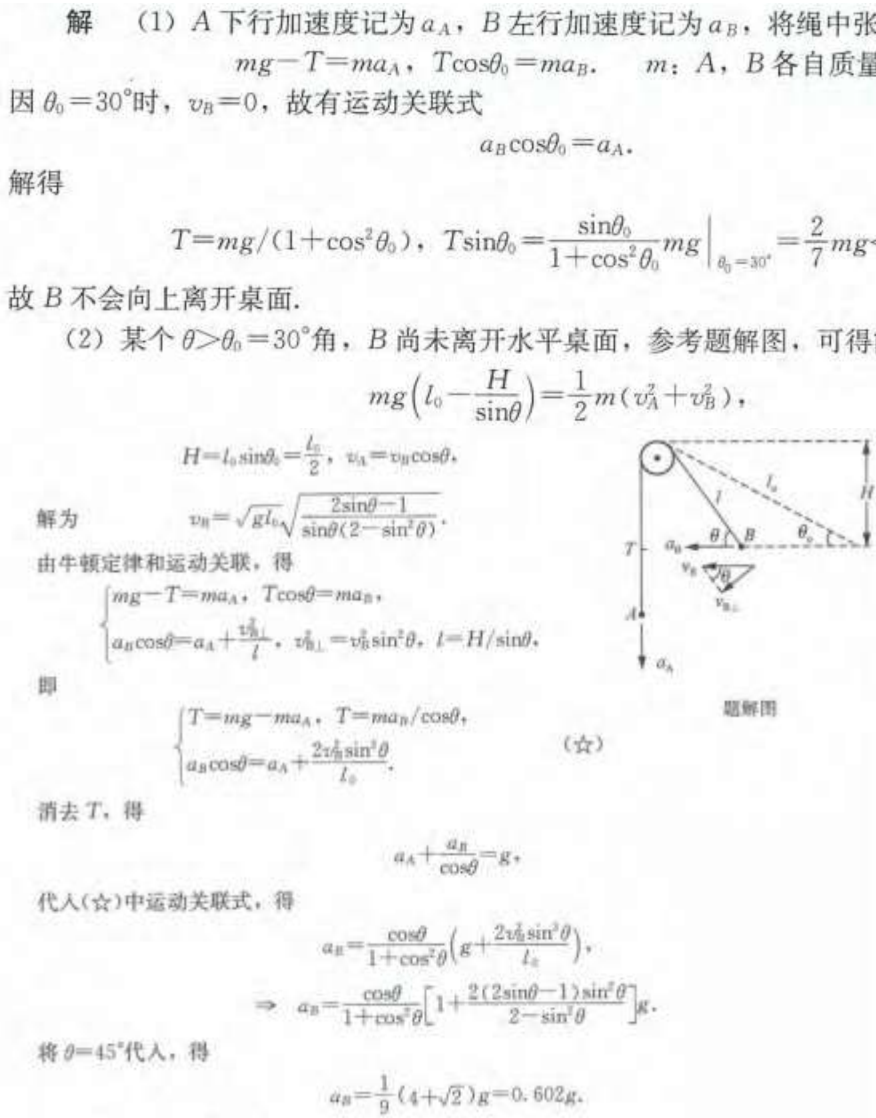
（1）通过计算，判断释放后一瞬间物块 B 是否会向上离开水平桌面？

（2）计算右侧绳段与水平桌面间夹角 $\theta>30^\circ$ 时物块 B 的左行加速度 a_B（假设此前 B 一直不会向上离开水平桌面），给出 $\theta=45^\circ$ 时的 a_B 值；

（3）试问右侧绳段与水平桌面夹角 θ（精确到 0.1°）达到什么值时，物块 B 会向上离开水平桌面．

Just do it

Solution



解 （1）A 下行加速度记为 a_A，B 左行加速度记为 a_B，将绳中张力记为 T，则有

$$mg-T=ma_A，\quad T\cos\theta_0=ma_B. \qquad m：A，B\text{ 各自质量}$$

因 $\theta_0=30°$时，$v_B=0$，故有运动关联式

$$a_B\cos\theta_0=a_A.$$

解得

$$T=mg/(1+\cos^2\theta_0)，\quad T\sin\theta_0=\frac{\sin\theta_0}{1+\cos^2\theta_0}mg\bigg|_{\theta_0=30°}=\frac{2}{7}mg<mg,$$

故 B 不会向上离开桌面.

（2）某个 $\theta>\theta_0=30°$角，B 尚未离开水平桌面，参考题解图，可得能量方程：

$$mg\left(l_0-\frac{H}{\sin\theta}\right)=\frac{1}{2}m(v_A^2+v_B^2),$$

$$H=l_0\sin\theta_0=\frac{l_0}{2},\quad v_A=v_B\cos\theta.$$

解为

$$v_B=\sqrt{gl_0}\sqrt{\frac{2\sin\theta-1}{\sin\theta(2-\sin^2\theta)}}.$$

由牛顿定律和运动关联，得

$$\begin{cases}mg-T=ma_A，\quad T\cos\theta=ma_B，\\ a_B\cos\theta=a_A+\dfrac{v_{B\perp}^2}{l}，\quad v_{B\perp}^2=v_B^2\sin^2\theta，\quad l=H/\sin\theta.\end{cases}$$

题解图

即

$$\begin{cases}T=mg-ma_A，\quad T=ma_B/\cos\theta，\\ a_B\cos\theta=a_A+\dfrac{2v_B^2\sin^2\theta}{l_0}.\end{cases} \qquad (☆)$$

消去 T，得

$$a_A+\frac{a_B}{\cos\theta}=g,$$

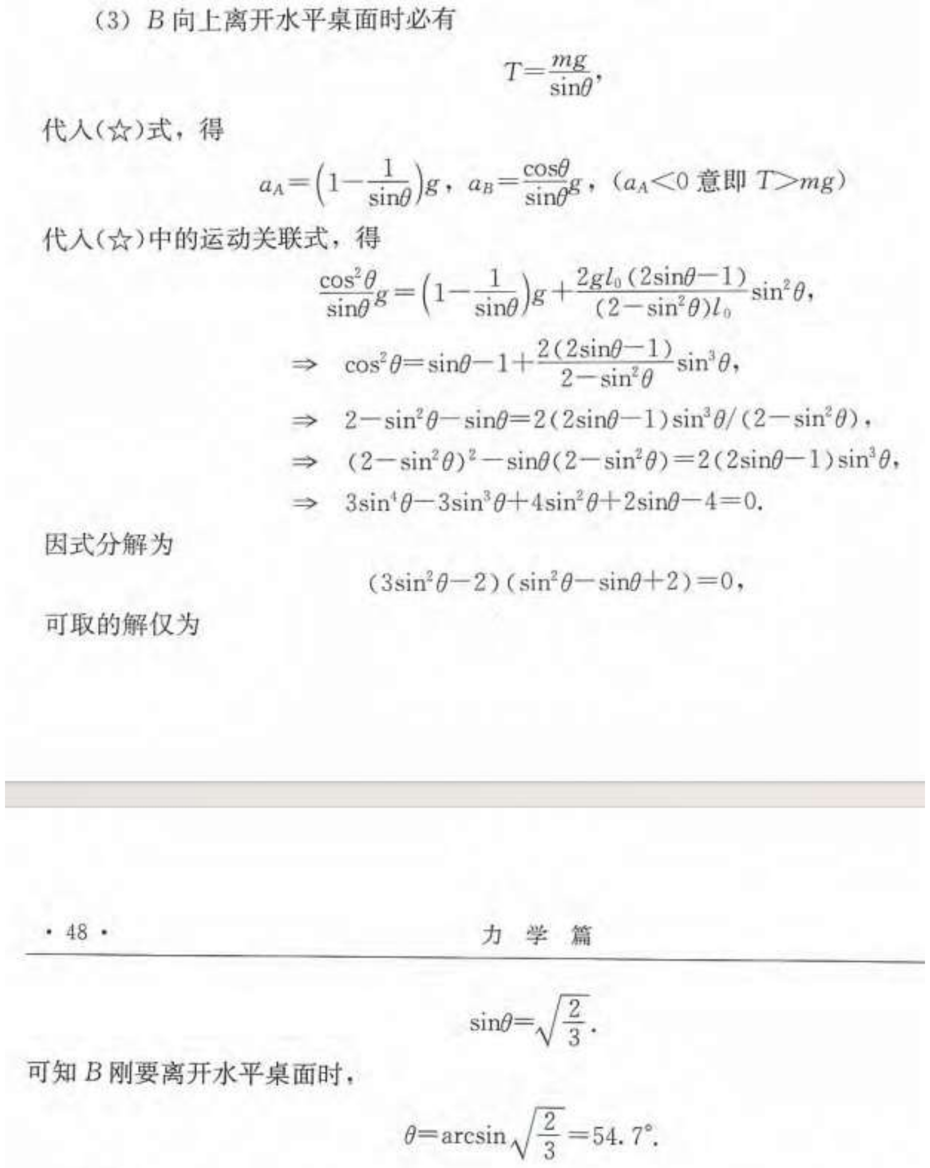
代入(☆)中运动关联式，得

$$a_B=\frac{\cos\theta}{1+\cos^2\theta}\left(g+\frac{2v_B^2\sin^2\theta}{l_0}\right),$$

$$\Rightarrow\quad a_B=\frac{\cos\theta}{1+\cos^2\theta}\left[1+\frac{2(2\sin\theta-1)\sin^2\theta}{2-\sin^2\theta}\right]g.$$

将 $\theta=45°$代入，得

$$a_B=\frac{1}{9}(4+\sqrt{2})g=0.602g.$$



（3）B 向上离开水平桌面时必有

$$T=\frac{mg}{\sin\theta},$$

代入(☆)式，得

$$a_A=\left(1-\frac{1}{\sin\theta}\right)g，\quad a_B=\frac{\cos\theta}{\sin\theta}g，\quad (a_A<0\text{ 意即 }T>mg)$$

代入(☆)中的运动关联式，得

$$\frac{\cos^2\theta}{\sin\theta}g=\left(1-\frac{1}{\sin\theta}\right)g+\frac{2gl_0(2\sin\theta-1)}{(2-\sin^2\theta)l_0}\sin^2\theta,$$

$$\Rightarrow\quad \cos^2\theta=\sin\theta-1+\frac{2(2\sin\theta-1)}{2-\sin^2\theta}\sin^3\theta,$$

$$\Rightarrow\quad 2-\sin^2\theta-\sin\theta=2(2\sin\theta-1)\sin^3\theta/(2-\sin^2\theta),$$

$$\Rightarrow\quad (2-\sin^2\theta)^2-\sin\theta(2-\sin^2\theta)=2(2\sin\theta-1)\sin^3\theta,$$

$$\Rightarrow\quad 3\sin^4\theta-3\sin^3\theta+4\sin^2\theta+2\sin\theta-4=0.$$

因式分解为

$$(3\sin^2\theta-2)(\sin^2\theta-\sin\theta+2)=0,$$

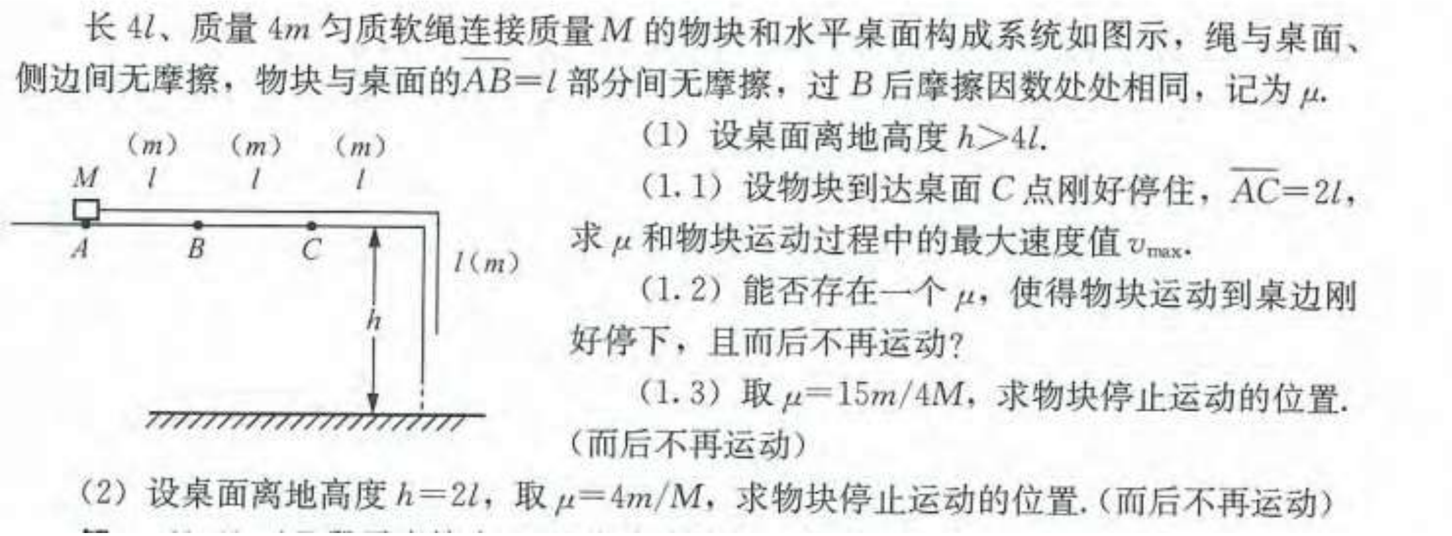
可取的解仅为

$$\sin\theta=\sqrt{\frac{2}{3}}.$$

可知 B 刚要离开水平桌面时，

$$\theta=\arcsin\sqrt{\frac{2}{3}}=54.7°.$$

绳子滑落模型



长 $4l$、质量 $4m$ 匀质软绳连接质量 M 的物块和水平桌面构成系统如图示，绳与桌面、侧边间无摩擦，物块与桌面的 $\overline{AB}=l$ 部分间无摩擦，过 B 后摩擦因数处处相同，记为 μ.

(1) 设桌面离地高度 $h>4l$.

(1.1) 设物块到达桌面 C 点刚好停住，$\overline{AC}=2l$，求 μ 和物块运动过程中的最大速度值 $v_{\max}$.

(1.2) 能否存在一个 μ，使得物块运动到桌边刚好停下，且而后不再运动?

(1.3) 取 $\mu=15m/4M$，求物块停止运动的位置.(而后不再运动)

(2) 设桌面离地高度 $h=2l$，取 $\mu=4m/M$，求物块停止运动的位置.(而后不再运动)

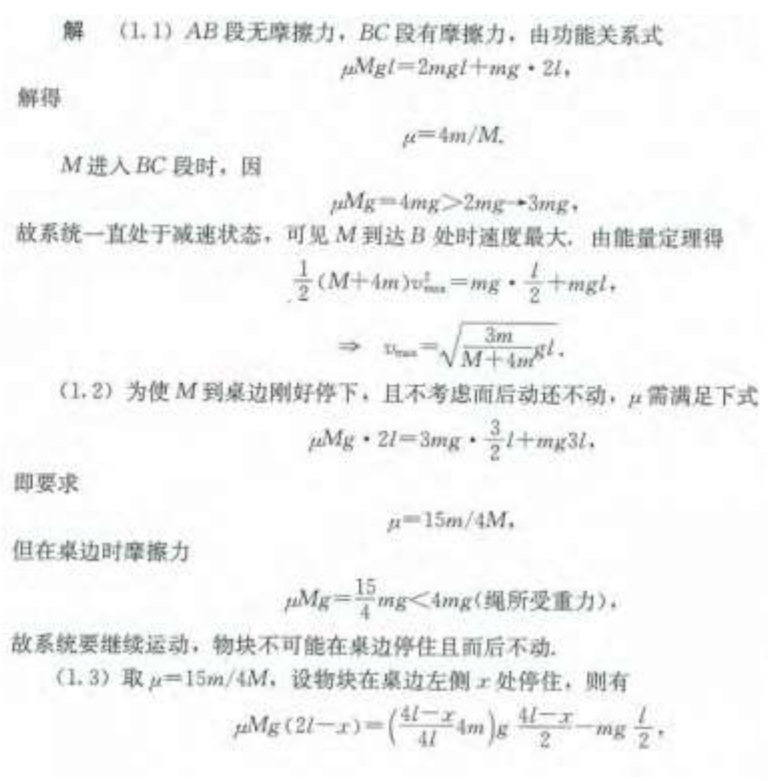
绳子下滑两种解法:

1：对一段微元的绳子做受力分析，得加速度

2：对整根绳子受力分析，再微分求dl的动力学变化

（不要忽略绳子一端固定时可能存在的拉力/支持力）

Solution：



解　（1.1）AB 段无摩擦力，BC 段有摩擦力，由功能关系式

$$\mu Mgl=2mgl+mg\cdot 2l,$$

解得

$$\mu=4m/M.$$

M 进入 BC 段时，因

$$\mu Mg=4mg>2mg-\cdot 3mg,$$

故系统一直处于减速状态，可见 M 到达 B 处时速度最大．由能量定理得

$$\frac{1}{2}(M+4m)v_{\max}^2=mg\cdot\frac{l}{2}+mgl,$$

$$\Rightarrow\quad v_{\max}=\sqrt{\frac{3m}{M+4m}gl}.$$

（1.2）为使 M 到桌边刚好停下，且不考虑而后动还不动，μ 需满足下式

$$\mu Mg\cdot 2l=3mg\cdot\frac{3}{2}l+mg3l,$$

即要求

$$\mu=15m/4M,$$

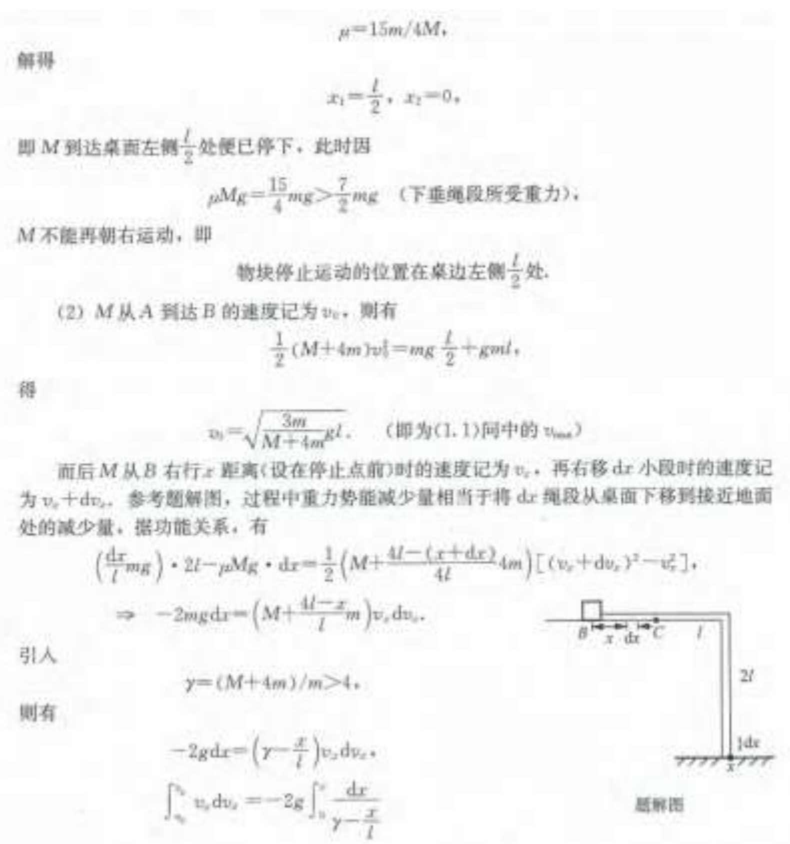
但在桌边时摩擦力

$$\mu Mg=\frac{15}{4}mg<4mg(\text{绳所受重力}),$$

故系统要继续运动，物块不可能在桌边停住且而后不动．

（1.3）取 $\mu=15m/4M$，设物块在桌边左侧 x 处停住，则有

$$\mu Mg(2l-x)=\left(\frac{4l-x}{4l}4m\right)g\,\frac{4l-x}{2}-mg\,\frac{l}{2},$$



$$\mu=15m/4M,$$

解得

$$x_1=\frac{l}{2},\ x_2=0,$$

即 M 到达桌面左侧$\frac{l}{2}$处便已停下，此时因

$$\mu Mg=\frac{15}{4}mg>\frac{7}{2}mg\quad(\text{下垂绳段所受重力}),$$

M 不能再朝右运动，即

物块停止运动的位置在桌边左侧$\frac{l}{2}$处．

（2）M 从 A 到达 B 的速度记为 v_0，则有

$$\frac{1}{2}(M+4m)v_0^2=mg\,\frac{l}{2}+gml,$$

得

$$v_0=\sqrt{\frac{3m}{M+4m}gl}.\quad(\text{即为(1.1)问中的 }v_{\max})$$

而后 M 从 B 右行 x 距离（设在停止点前）时的速度记为 v_x，再右移 dx 小段时的速度记为 v_x+dv_x．参考题解图，过程中重力势能减少量相当于将 dx 绳段从桌面下移到接近地面处的减少量，据功能关系，有

$$\left(\frac{dx}{l}mg\right)\cdot 2l-\mu Mg\cdot dx=\frac{1}{2}\left(M+\frac{4l-(x+dx)}{4l}4m\right)[(v_x+dv_x)^2-v_x^2],$$

$$\Rightarrow\quad -2mg\,dx=\left(M+\frac{4l-x}{l}m\right)v_x\,dv_x.$$

题解图

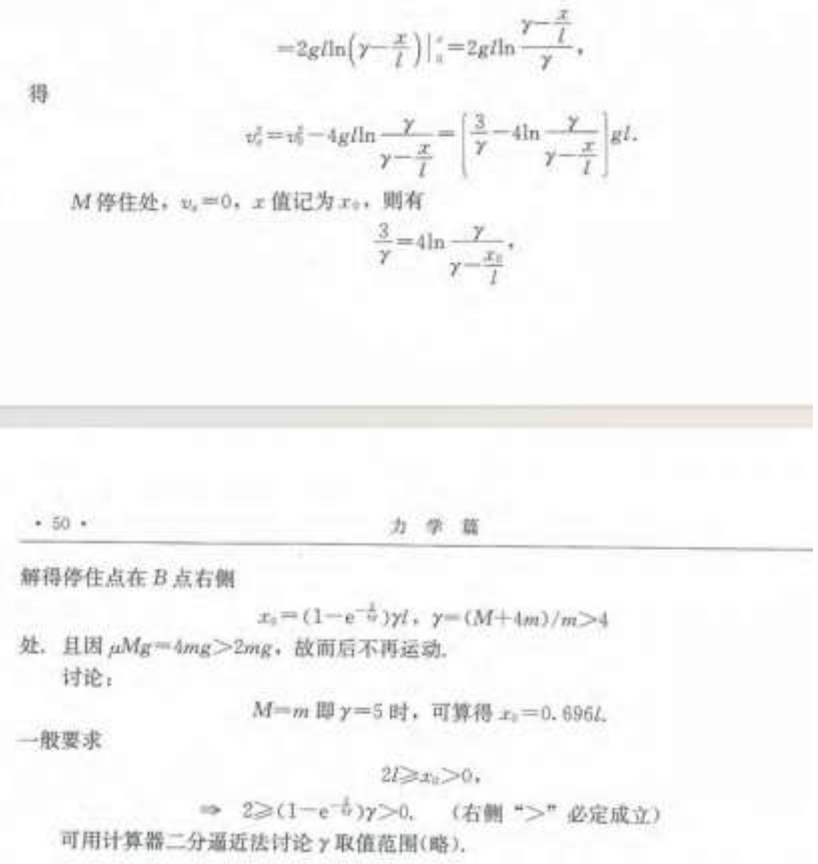
引入

$$\gamma=(M+4m)/m>4,$$

则有

$$-2g\,dx=\left(\gamma-\frac{x}{l}\right)v_x\,dv_x,$$

$$\int_{v_0}^{v_x}v_x\,dv_x=-2g\int_0^x\frac{dx}{\gamma-\frac{x}{l}}$$



$$=2gl\ln\left(\gamma-\frac{x}{l}\right)\Big|_0^x=2gl\ln\frac{\gamma-\frac{x}{l}}{\gamma},$$

得

$$v_x^2=v_0^2-4gl\ln\frac{\gamma}{\gamma-\frac{x}{l}}=\left[\frac{3}{\gamma}-4\ln\frac{\gamma}{\gamma-\frac{x}{l}}\right]gl.$$

M 停住处，$v_x=0$，x 值记为 x_0，则有

$$\frac{3}{\gamma}=4\ln\frac{\gamma}{\gamma-\frac{x_0}{l}},$$

解得停住点在 B 点右侧

$$x_0=(1-e^{-\frac{3}{4\gamma}})\gamma l,\ \gamma=(M+4m)/m>4$$

处．且因 $\mu Mg=4mg>2mg$，故而后不再运动．

讨论：

$M=m$ 即 $\gamma=5$ 时，可算得 $x_0=0.696l$.

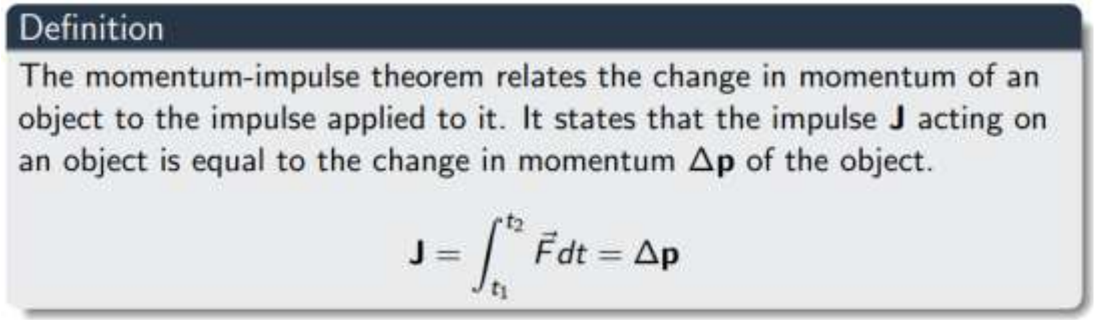
一般要求

$$2l\geqslant x_0>0,$$

$$\Rightarrow\quad 2\geqslant(1-e^{-\frac{3}{4\gamma}})\gamma>0.\quad(\text{右侧“}>\text{”必定成立})$$

可用计算器二分逼近法讨论 γ 取值范围（略）．

Momentum-Impulse Theorem



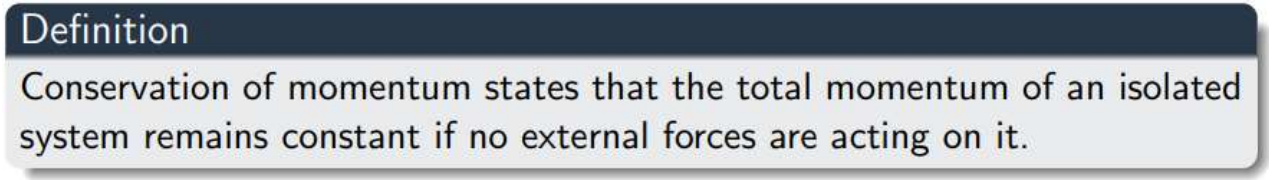
Definition

The momentum-impulse theorem relates the change in momentum of an object to the impulse applied to it. It states that the impulse $\mathbf{J}$ acting on an object is equal to the change in momentum $\Delta\mathbf{p}$ of the object.

$$\mathbf{J} = \int_{t_1}^{t_2} \vec{F} dt = \Delta\mathbf{p}$$

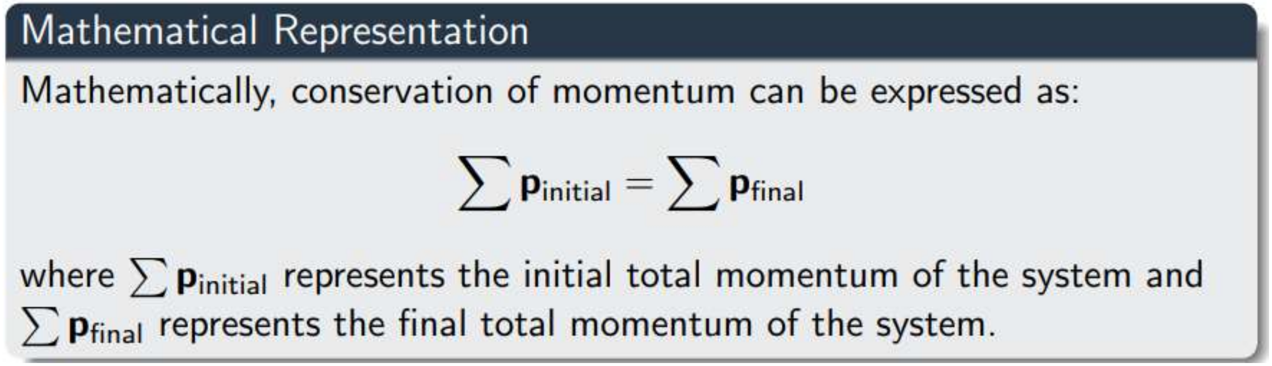
$$F = ma = m\frac{dv}{dt} = \frac{dp}{dt}$$

Conservation of Momentum



Definition

Conservation of momentum states that the total momentum of an isolated system remains constant if no external forces are acting on it.



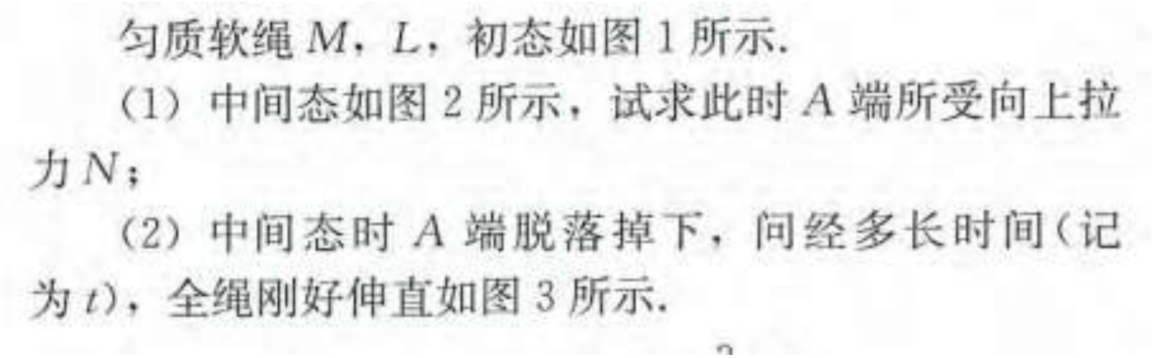
Mathematical Representation

Mathematically, conservation of momentum can be expressed as:

$$\sum \mathbf{p}_{\text{initial}} = \sum \mathbf{p}_{\text{final}}$$

where $\sum \mathbf{p}_{\text{initial}}$ represents the initial total momentum of the system and $\sum \mathbf{p}_{\text{final}}$ represents the final total momentum of the system.

Simple Question



匀质软绳 M，L，初态如图 1 所示.

（1）中间态如图 2 所示，试求此时 A 端所受向上拉力 N；

（2）中间态时 A 端脱落掉下，问经多长时间（记为 t），全绳刚好伸直如图 3 所示.

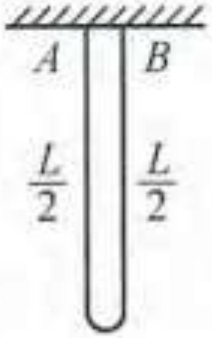


图1

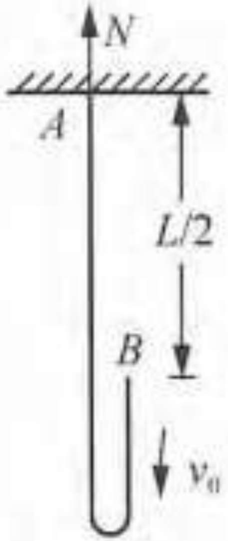
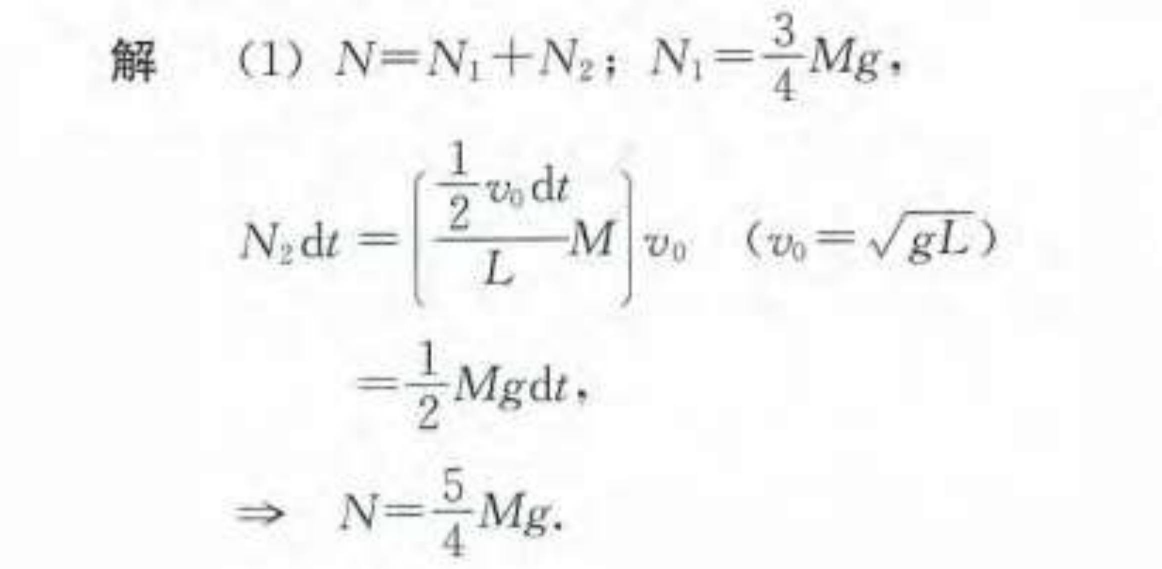


图2

图3

Solution

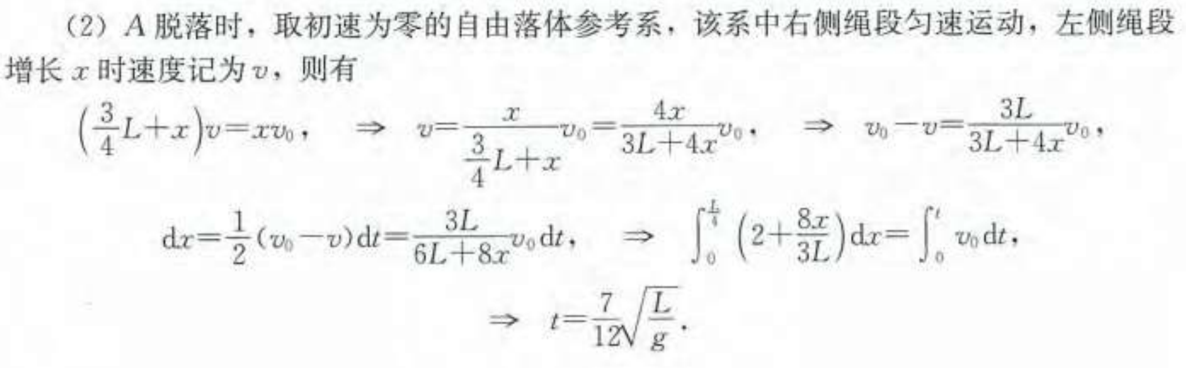


解　（1）$N=N_1+N_2$；$N_1=\frac{3}{4}Mg$，

$$N_2\mathrm{d}t=\left[\frac{\frac{1}{2}v_0\mathrm{d}t}{L}M\right]v_0\quad(v_0=\sqrt{gL})$$

$$=\frac{1}{2}Mg\mathrm{d}t,$$

$$\Rightarrow\quad N=\frac{5}{4}Mg.$$



（2）A 脱落时，取初速为零的自由落体参考系，该系中右侧绳段匀速运动，左侧绳段增长 x 时速度记为 v，则有

$$\left(\frac{3}{4}L+x\right)v=xv_0,\quad\Rightarrow\quad v=\frac{x}{\frac{3}{4}L+x}v_0=\frac{4x}{3L+4x}v_0,\quad\Rightarrow\quad v_0-v=\frac{3L}{3L+4x}v_0,$$

$$\mathrm{d}x=\frac{1}{2}(v_0-v)\mathrm{d}t=\frac{3L}{6L+8x}v_0\mathrm{d}t,\quad\Rightarrow\quad\int_0^{\frac{L}{4}}\left(2+\frac{8x}{3L}\right)\mathrm{d}x=\int_0^t v_0\mathrm{d}t,$$

$$\Rightarrow\quad t=\frac{7}{12}\sqrt{\frac{L}{g}}.$$

The complex questions continue ……

Thanks

From Fallout 4