Complex Analysis

Complex Analysis

Author: Linda Wei, Nyakkeru

Department UMJI,PKU

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With great gratitude to my friend Nyakkeru in Osu! Mania.

1 Preliminary

1.1 Holomophic Function

We say that a function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, or **holomorphic**, at $z \in \mathbb{C}$ if

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, h \in \mathbb{C}$$

A function is *holomorphic* on an open set $\Omega \subset \mathbb{C}$ if it is holomorphic at every $z \in \Omega$. A function that is holomorphic on C is called **entire**.

1.1.1 Example from Stein

eg. Any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is holomorphic on $\mathbb{C}$ (i.e., entire) and has derivative

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

1.2 Cauchy Riemann Equations

1.2.1 Cauchy Riemann on Cartesian Coordinate

If f is holomorphic, $f = u(x, y) + iv(x, y)$ the component functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof:

$$\begin{aligned}
f'\left(z_0\right) &= \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f\left(z_0+h\right)-f\left(z_0\right)}{h} = \lim_{\substack{h_1 \to 0 \\ h_1 \in \mathbb{R}}} \frac{f\left(x_0+h_1+i y_0\right)-f\left(x_0+i y_0\right)}{h_1} \\
&= \lim_{\substack{h_1 \to 0 \\ h_1 \in \mathbb{R}}} \frac{u\left(x_0+h_1, y_0\right)-u\left(x_0, y_0\right)}{h_1} + i \lim_{\substack{h_1 \to 0 \\ h_1 \in \mathbb{R}}} \frac{v\left(x_0+h_1, y_0\right)-v\left(x_0, y_0\right)}{h_1} \\
&= \left.\frac{\partial u}{\partial x}\right|_{\left(x_0, y_0\right)} + i \left.\frac{\partial v}{\partial x}\right|_{\left(x_0, y_0\right)}
\end{aligned}$$

$$\begin{aligned}
f'\left(z_0\right) &= \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f\left(z_0+h\right)-f\left(z_0\right)}{h} = \lim_{\substack{h_2 \to 0 \\ h_2 \in \mathbb{R}}} \frac{f\left(x_0+i\left(y_0+h_2\right)\right)-f\left(x_0+i y_0\right)}{i h_2} \\
&= -i \left.\frac{\partial u}{\partial y}\right|_{\left(x_0, y_0\right)} + \left.\frac{\partial v}{\partial y}\right|_{\left(x_0, y_0\right)}
\end{aligned}$$

1.2.2 Cauchy Riemann on Polar Coordinate

$$\frac{\partial v}{\partial \theta} \cdot \frac{1}{r} = \frac{\partial u}{\partial r}, -\frac{\partial u}{\partial \theta}\frac{1}{r} = \frac{\partial v}{\partial r}$$

Proof:

$$\text{since } d(re^{i\theta}) = dr * e^{i\theta} + r * e^{i\theta}\delta\theta$$

Radical approaches to zero: $f'(z) = \lim_{\substack{h\to 0\\ hoc}} \frac{f(z_0+h)-f(z_0)}{h}$

$$\begin{aligned}
&= \lim_{\Delta re^{i\theta}\to 0} \frac{u(r+\Delta r,\theta) + iv(r+\Delta r,\theta) - n(r,\theta) - iv(r,\theta)}{\Delta r \cdot e^{i\theta}} \\
&= \lim_{\sim r} \frac{n(r+\Delta v,\theta) - u(r,\theta)}{\Delta re^{i\theta}} + i\frac{v(r+\Delta r,\theta) - v(r,\theta)}{\Delta re^{i\theta}} \\
&= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) e^{-i\theta}
\end{aligned}$$

Tangential approaches to zero: $f'(z) = \lim_{\substack{h\to 0\\ h\in c}} \frac{f(z_0+h)-f(z_0)}{h}$

$$\begin{aligned}
&= \lim_{r\cdot e^{i\theta}\Delta\theta\to 0} \frac{n(r,\theta+\Delta\theta) + iv(r,\theta+\Delta\theta) - u(r,\theta) - iv(r,\theta)}{ire^{i\theta}\cdot\Delta\theta.} \\
&= \text{lin} \frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{ir\Delta\theta \cdot e^{i\theta}} + i\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{ir\Delta\theta e^{i\theta}} \\
&= \left(\frac{\partial v}{\partial \theta}\cdot\frac{1}{r} - i\frac{1}{r}\frac{\partial u}{\partial \theta}\right) e^{-i\theta} \\
&\therefore \frac{\partial v}{\partial \theta} \cdot \frac{1}{r} = \frac{\partial u}{\partial r}, -\frac{\partial u}{\partial \theta}\frac{1}{r} = \frac{\partial v}{\partial r}.\text{Q.E.D}
\end{aligned}$$

1.2.3 Cauchy Riemann Equations and Holomophicity

My friend Nyakkeru has conclude for you:

- Holomophic → Cauchy Riemann equations
- Cauchy Riemann + Partial derivatives continuous → Holomophic.

1.2.4 Collary of Cauchy Riemann Equation

We define

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{\partial}{i\partial y})$$

and it follows that

$$\frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}, \frac{\partial f}{\partial \bar{z}} = 0$$

2 Integration along curve on Complex Plane

2.1 Definitions and Notations

A parametrized curve is a set $\ell \subset \mathbb{C}$ such parametrization $\gamma\colon I \to \mathcal{C}$ for some interval $I \to \mathbb{C}$, where γ is locally smooth if there exists a parametrization γ (t) $\gamma'(t) \neq 0$ for all $t \in I$.

We borrow from $\mathbb{R}^2$ the concept of *positive* curves, i.e., those parametrized in a counterfashion, respectively.

Definition. Let $\Omega \subset \mathbb{C}$ be an open set, f holomorphic on Ω and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve. We then define the integral of f along $\mathcal{C}^*$ by

$$\int_{\mathcal{C}^*} f(z)dz := \int_I f(\gamma(t)) \cdot \gamma'(t)dt$$

Remark. Compare this with the definition of integrals of scalar functions and vector fields in, e.g., $\mathbb{R}^2$:

$$\int_{\mathcal{C}^*} f dr = \int_I f(\gamma(t)) \cdot |\gamma'(t)|\, dt, \quad \int_{\mathcal{C}^*} \langle F, d\vec{r} \rangle = \int_I \langle F(\gamma(t)), \gamma'(t) \rangle\, dt$$

Definition. Let $\Omega \subset \mathbb{C}$ be A an open set, that! a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \Omega$

2.2 Newton formula in complex plane

Theorem. If a continuous function f has a primitive F in Ω, and isa curve in Ω that begins at w_1 and ends at w_2, then

$$\int_{\mathcal{C}^*} f(z)\, dz = F(w_2) - F(w_1).$$

If ℓ is a closed curve in an open set Ω, and f is **continuous and has a primitive(Nyakkeru: This condition is not weaker than holomophic, this condition is a kind of Integrability instead of differentiable. But actually continuous itself is much weaker than holomophic)**, then,

$$\oint_{\mathcal{C}} f(z)dz = 0.$$

3 Property of Holomophic Functions

3.1 Steps to Cauchy Theorem

- Goursat's Theorem. Let $\Omega \subset \mathbb{C}$ be open and f holomorphic on Ω. Let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Then

  $$\oint_T f(z)dz = 0.$$

  however I think you do not need to understand it.

- Corollary of Goursat's Theorem. If f is holomorphic in an open an open set Ω that contains a rectangle R and its interior, then

  $$\oint_R f(z)dz = 0$$

- Cauchy Theorem(**Very important**) If f is holomorphic in a disc, then

  $$\oint_C f(z)dz = 0$$

  for any closed curve ℓ in that disc.

3.2 Cauchy Integral Formula

3.2.1 Theorem

Cauchy's Integral Formula. Suppose f is a holomorphic function in an open set $\Omega \subset \mathbb{C}$. If D is an open disc whose closure is contained in Ω, then

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}d\zeta \quad \text{for all } z \in D,$$

where $C = \partial D$ is the (positively oriented) boundary circle of D.

3.2.2 Explanation

Remark: This section is provided by Nyakkeru, it is written to give students in ECE a simple insight on this theorem rather than mathematical proof.

By Cauchy Theorem, we have

$$\oint_{\mathcal{C}} f(z)dz = 0$$

now we consider if the function $f(z)$ is not holomophic but almost holomophic, i.e. not holomophic at some points. For example, we definitely know that $\frac{f(z)}{z-z_0}$ has some problems in z_0. We call these points like z_0 **sigularities**.

Singularities can be classified into the following three types: the formal definition should consider **Analytic Continuation** which is not so necessary for you to master. Here provides simple methods to know what are they:

- Removable Singularity:z_0 such that $\lim_{z\to z_0} f(z) = 0$
- Pole:z_0 such that $\lim_{z\to z_0} f(z) = \infty$
- Essential Singularity: not Removable Singularity or Pole.

Now I give you insight on how to derive the above theorem. We first consider

$$\oint_{\mathcal{C}} \frac{1}{z}dz = 2\pi i, z_0 = 0$$

in which $\mathcal{C}$ is a closed curve contains z_0.

Why it happens, we suppose C is an unit cycle, i.e.

$$\oint_{|z|=1} \frac{1}{z}dz$$

we can change variable using $z = e^{i\theta}$ and we get

$$\int_0^{2\pi} e^{-i\theta}de^{i\theta} = 2\pi i$$

Then, how about closed curve which is not unit circle. Here I will explain why the integral along those curves are the same as the unit circle. By adding a path (red line) to connect the two curve contain the origin(pole in this case) and draw some arrows on the ugly pink and blue circle provided by aa. We can find that the pink and blue curve together with the red path will form a closed curve and the function f is now holomophic on the region formed by the closed curve. Thus if we use this route to integrate, the answer thould be zero.

That is: counterclockwise pink(positive pink)+path(into origin)+clockwise blue(negative blue)+path(out of origin) is zero.

That is: positive pink + negative blue is zero.

That is: pink is blue.

Here, we have shown that:

$$\oint_{\mathcal{C}} \frac{1}{z}dz = 2\pi i$$

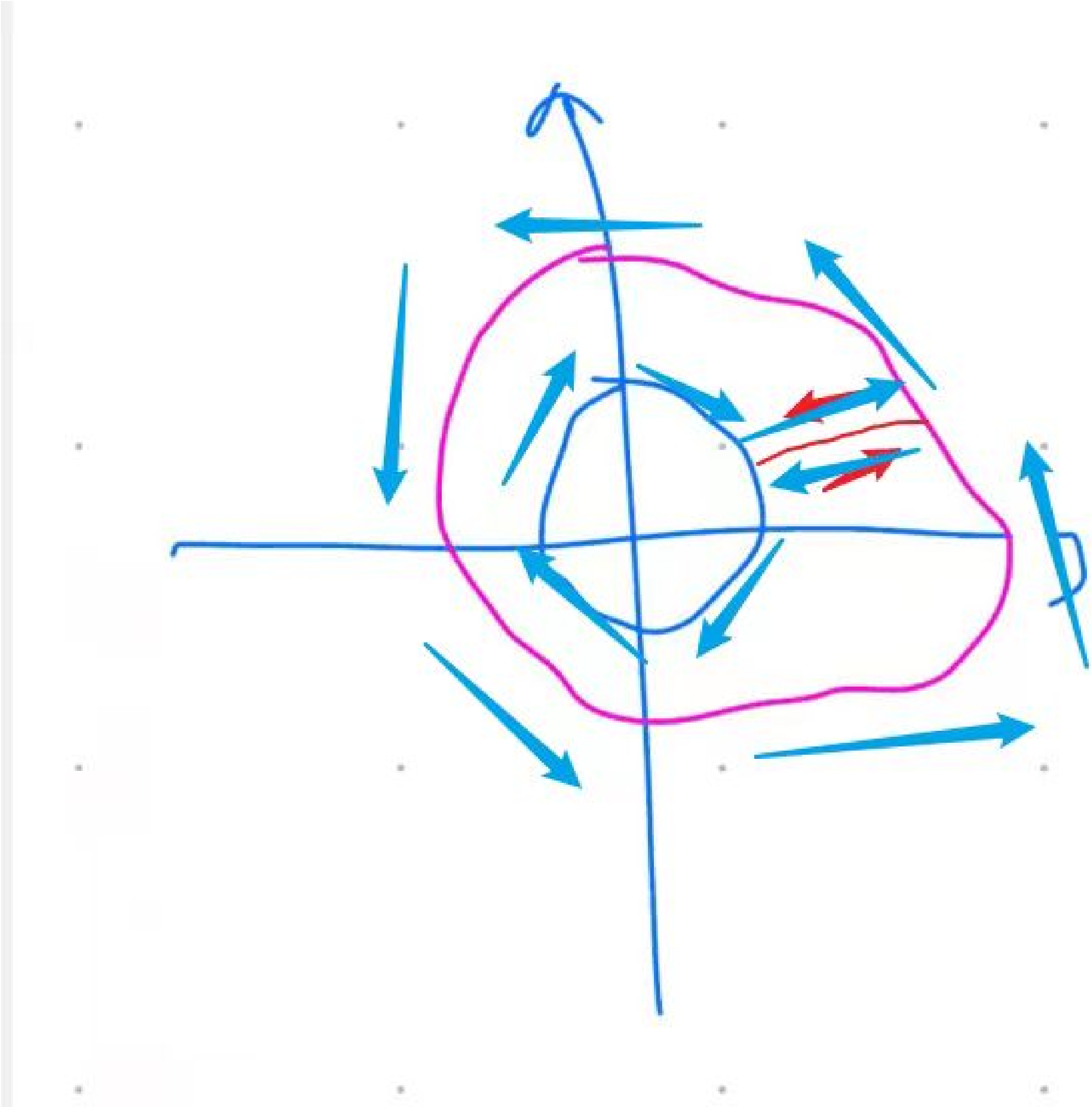

Figure 1: Explain about the last statement

Now we consider

$$\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } z \in D,$$

we can consider the integral near the neibor of z to avoiding reaching z.

$$\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = \oint_{C_\epsilon} \frac{f(z)}{\zeta - z} d\zeta = f(z) 2\pi i$$

then we get

$$f(z) = \frac{1}{2\pi i} \frac{f(z)}{\zeta - z} d\zeta$$

very easily without mathematical analysis. And we perform n-th order derivative, we get

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

4 Laurent Expansion

A complex function can be Taylor expanded as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, \mathrm{d}\xi$$

But this requires the function to be holomorphic within the circular domain

$$|z - z_0| < R$$

Therefore, for the annular domain

$$R_1 < |z - z_0| < R_2$$

, we have the bilateral power series expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, \mathrm{d}\xi$$

This is known as the Laurent series (the positive power series is called the holomorphic part, with a convergence domain of

$$|z - z_0| < R_2$$

; the negative power series is called the principal part, with a convergence domain of

$$|z - z_0| > R_1$$

).

Using the Laurent expansion,

$$\oint_\gamma f(z)\mathrm{d}z = \sum_{k=-\infty}^{\infty} c_k \oint_{\gamma_0} (z - z_0)^k \ \mathrm{d}z$$

Only the term with

$$k = -1$$

is non-zero. Therefore, we give the coefficient of the term with power

$$-1$$

a special name "residue", denoted as

$$\operatorname{Res}\left[f(z), z_0\right]$$

, so the above equation transforms into

$$c_{-1} \oint_{\gamma_0} (z - z_0)^{-1} \ \mathrm{d}z = 2\pi \mathrm{i} c_{-1} = 2\pi \mathrm{i} \operatorname{Res}\left[f(z), z_0\right]$$

.

The residue at infinity is defined as

$$\operatorname{Res}[f(z), \infty] = -c_{-1}$$

Nyakkeru further gives two insight on why only $k = -1$ contribute to the integral.

- You can perform the integral on $z^{-n} n \neq 1$ along unit circle
- $1/z$ is the conjugate of z. You can consider the relationship between dz and $1/z$ as vector, their product will be a pure imaginary number i, runs along the circle 2π and get 2πi,but if we divide another z, we will introduce rotation, the integral will be cancelled out to be zero.

5 Residue Calculus

5.1 Residue Theorem

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi \mathrm{i} \sum_{k=1}^{n} \operatorname{Res}\left[f(z), z_k\right]$$

is residue theorem.

5.2 Laurent Expansion near m-th order pole

$$f(z)=\frac{c_{-m}}{(z-z_0)^m}+\frac{c_{-m+1}}{(z-z_0)^{m-1}}+\cdots+\frac{c_{-1}}{z-z_0}+c_0+c_1(z-z_0)+\cdots$$

5.3 Residue of m-th order pole

$$\operatorname{Res}\left[f(z),z_0\right]=\frac{1}{(m-1)!}\left[(z-z_0)^m f(z)\right]^{(m-1)}\Big|_{z=z_0}$$

5.4 Jordan Lemma

If $f(x)$ is continuous on $\{z|r\le|z|<+\infty,\operatorname{Im}z\ge0\}$ $\lim_{z\to\infty}f(z)=0$,for all $\alpha>0$, we have

$$\lim_{R\to+\infty}\int_{\gamma_R}f(z)\mathrm{e}^{\mathrm{i}\alpha z}\mathrm{d}z=0$$

in which $\gamma_R=\{R\mathrm{e}^{\mathrm{i}\theta}|0\le\theta\le\pi\},R>r$

5.5 Large Arc Lemma

If $\lim_{z\to\infty}(z-z_0)f(z)=A$,then

$$\lim_{R\to+\infty}\int_{\gamma_R}f(z)\mathrm{d}z=\mathrm{i}A(\theta_2-\theta_1)$$

5.6 Small Arc Lemma

If $\lim_{z\to0}(z-z_0)f(z)=A$,then

$$\lim_{\delta\to0}\int_{\gamma_\delta}f(z)\mathrm{d}z=\mathrm{i}A(\theta_2-\theta_1)$$

5.7 Form of $\int_0^{2\pi} f(cosx, sinx)dx$

Evaluate

$$\int_0^{2\pi} \frac{1}{1+acosx}dx$$

In this kind of form, f is continuous function of $sinx$ and $cosx$.

5.8 Form of $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$

Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}$$

In this case,$q(x)$ always doesn't admit real roots(actually we can deal with the case with real root, but it is too hard for you.) And, $q(x)$ is always "higher" than $p(x)$ by at least 2 orders.

5.9 Form of $\int_0^{+\infty} f(x) \cos mx \mathrm{d}x$

Evaluate

$$I = \int_0^{+\infty} \frac{\cos bx}{x^2+a^2}\mathrm{d}x$$

In this case,$f(x)$ is always even functions, and the denominator always "higher" then numerator for at least 1 order.

5.10 Form of real first order pole

In this case, we should use Small Arc Lemma.
Evaluate the Dirichlet Integral:

$$\int_{\infty}^{\infty} \frac{sinx}{x}dx$$

5.11 Remark

There are many other cases... But these are four cases that can solve most of the problems in HMWK4. If Runze tests other cases which need you to construct complex contour, just **give up**

6 Reference

1. Complex Analysis Stein 2. VV286 lecture slide Horst